応用数学　A

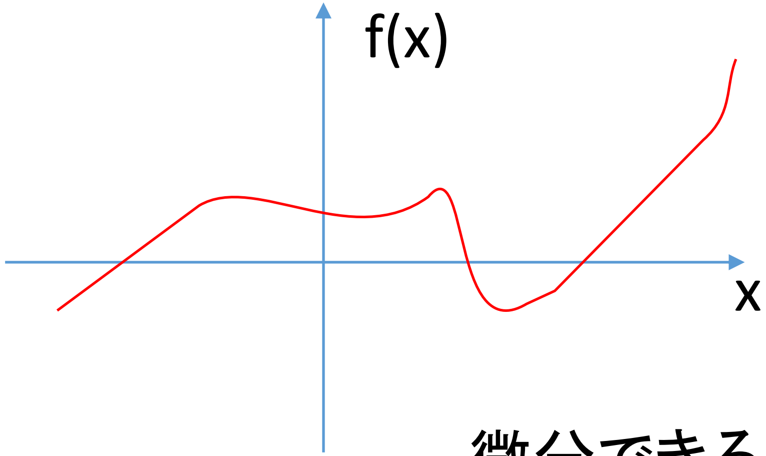
複素数の微分

$$f'(x) = \frac{df(x)}{dx}$$

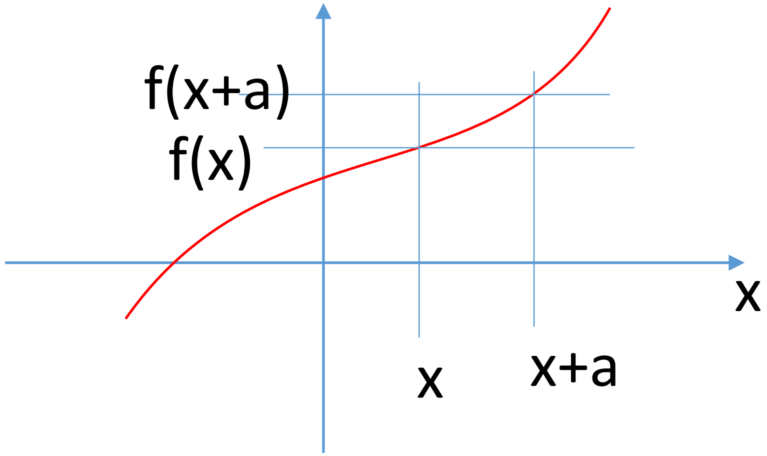
$$z = x + iy$$

$$f'(z)???$$

微分



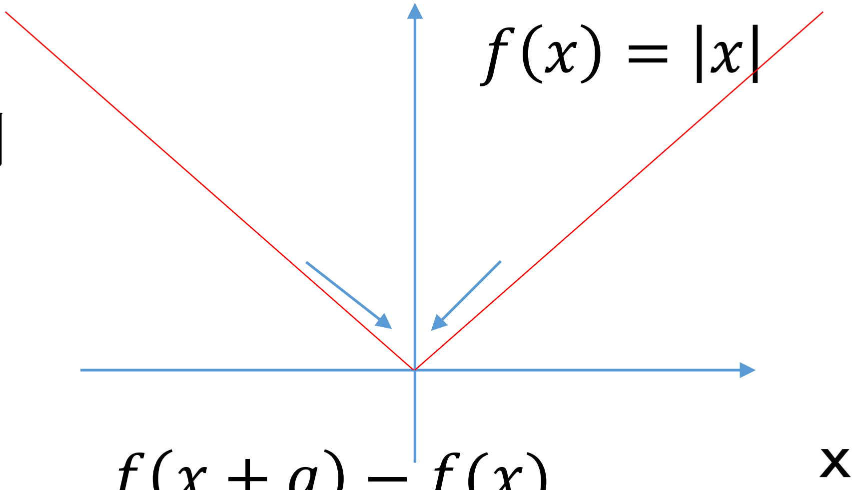
微分できる？



$$\lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

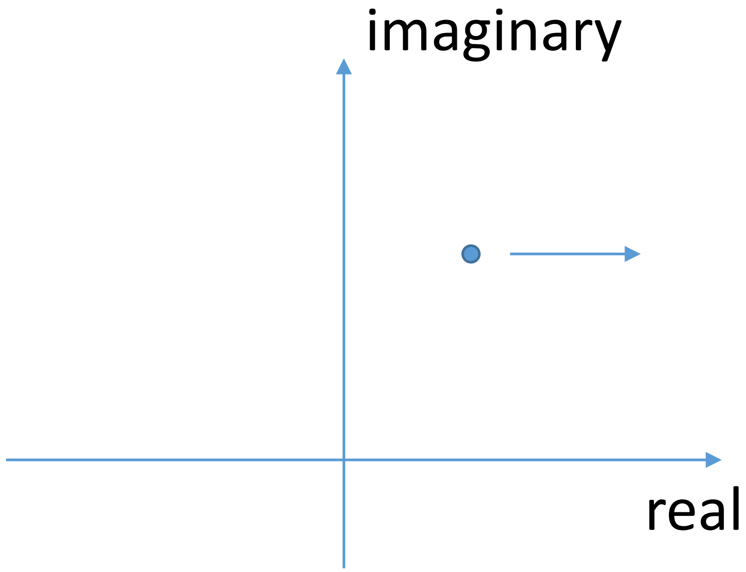
が存在するならば、微分可能

だめな代表例



$$\lim_{a \to +0} \frac{f(x+a) - f(x)}{a} \neq \lim_{a \to -0} \frac{f(x+a) - f(x)}{a}$$

複素数では？ $z = x + iy \quad w = f(z) = u(x,y) + iv(x,y)$ 複素数の関数



複素数での増減

$$\Delta w = f(z + \Delta z) - f(z) \qquad \Delta z = \Delta x + \Delta y$$

実部の増減

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x,y)$$

虚部の増減

$$\Delta v = v(x + \Delta x, y + \Delta y) - v(x,y)$$

1. 変化がReal に沿った場合 $\Delta z = \Delta x$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2. 変化がImaginary に沿った場合 $\Delta z = \Delta y$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y} \rightarrow \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

であるならば、微分可能　正則

コーシー・リーマンの条件

例：$f(z) = z^3$ は複素空間 $z = x + iy$ で微分可能（正則）か？

$$w = f(z) = u(x, y) + iv(x, y) = (x + iy)^3 = \underbrace{x^3 - 3xy^2}_{u(x,y)} + i(\underbrace{3x^2 y - y^3}_{v(x,y)})$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 - 3y^2 + i(6xy)$$

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - i(-6xy)$$

互いに等しい　　正則（微分可能）

$z = x + iy \quad w = f(z) = u(x,y) + iv(x,y)$ 複素数の関数

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

コーシー・リーマンの条件

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}、\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

したがって $\frac{\partial}{\partial x}$ して

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

$\Delta u = 0$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial^2 v}{\partial y^2}$$

$\Delta v = 0$

正則な複素関数の実部、虚部の関数は共役調和関数

例

$u(x,y) = x^2 - y^2 + x$ とする。

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2, \ \triangle u = 0$$ 調和関数

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ の関係（C-R）にある v(x, y)を探す。

$$\frac{\partial v}{\partial x} = -(-2y) = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

$$v = 2xy + p(y) \quad \frac{\partial v}{\partial y} = 2x + \frac{\partial p(y)}{\partial y} = 2x + 1 \quad p(y) = y + C \quad v = 2xy + y + C$$

共役調和関数

- このuとvで複素関数(u+vi)を作ると正則な複素関数になる。
- uとvが調和関数であってもu+viが正則とは限らない

例（続き）

$u(x,y)=x^2-y^2+x$ とする。

$\frac{\partial u}{\partial x}=2x+1,\quad \frac{\partial^2 u}{\partial x^2}=2,\quad \frac{\partial u}{\partial y}=-2y,\quad \frac{\partial^2 u}{\partial y^2}=-2,\ \triangle u=0$ 調和関数

$v(x,y)=2xy\quad \frac{\partial v}{\partial x}=2y\quad \frac{\partial^2 y}{\partial x^2}=0\quad \frac{\partial v}{\partial y}=2x\quad \frac{\partial^2 v}{\partial y^2}=0\quad \triangle v=0$ 調和関数

$u(x,y)+iv(x,y)$

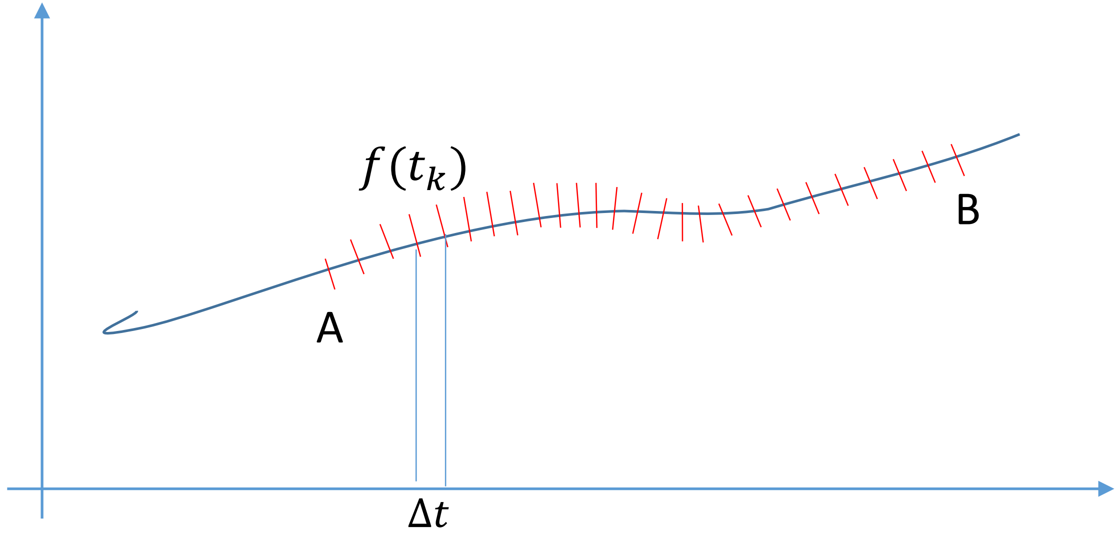
$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=(2x+1)+i(2y)$$

$$\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=(2x)-i(-2y)=(2x)+i(2y)$$

正則ではない

- uとvが調和関数であってもu+viが正則とは限らない

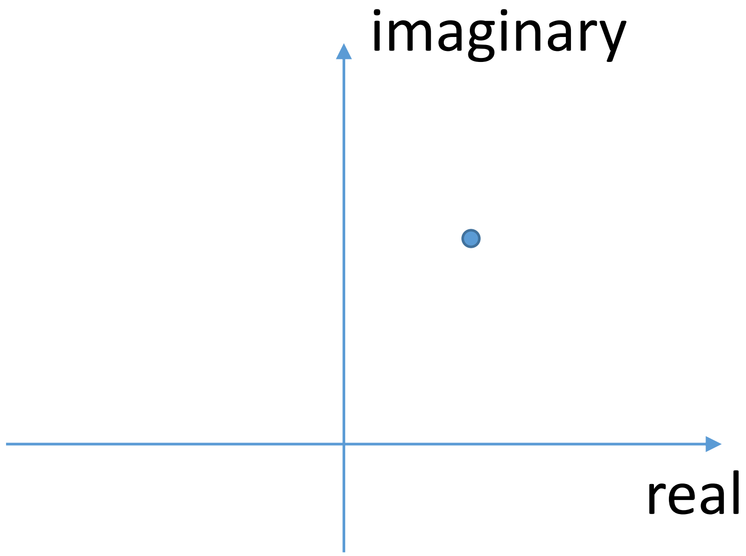
複素関数の積分

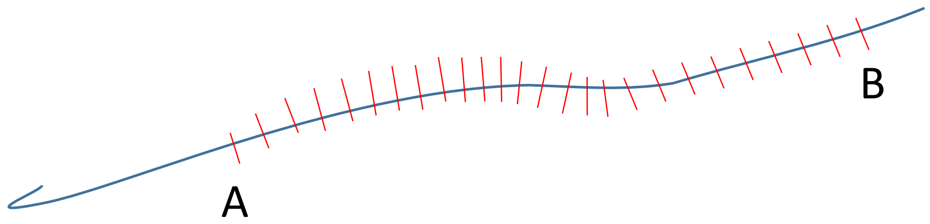


積分　微小区間 Δt　　$\int f(t)dt = \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f(t_k)\Delta t_k$

複素空間での積分

積分 $\int f(t)dt = \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} f(t_k)\Delta t_k$





複素数での増減 $\Delta t = \Delta x + \Delta y i$

$z = x + iy \quad w = f(z) = u(x,y) + iv(x,y)$

$$\int_{AB} f(z)dz = \lim_{\Delta \to 0} \sum_k f(z_k)\Delta z_k = \lim_{\Delta \to 0} \sum_k [u(x_k, y_k) + iv(x_k, y_k)](\Delta x_k + i\Delta y_k)$$

$$= \lim_{\Delta \to 0} \sum_k [u(x_k, y_k)\Delta x_k - v(x_k, y_k)\Delta y_k] + i \lim_{\Delta \to 0} \sum_k [v(x_k, y_k)\Delta x_k + u(x_k, y_k)\Delta y_k]$$

$$= \int_{AB} udx - vdy + i \int_{AB} vdx + udy$$

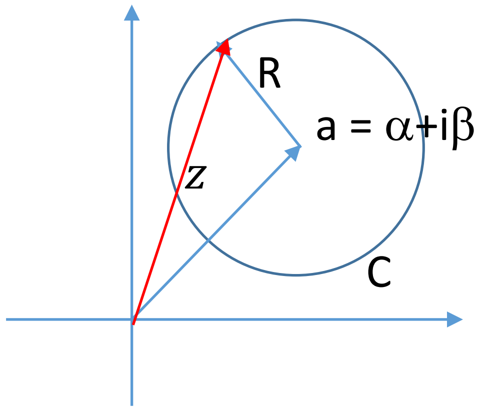
$x(t),\quad y(t)$ と考える $z(t) = x(t) + iy(t)$

$$\int_{AB} f(z)dz = \int_{AB} udx - vdy + i\int_{AB} vdx + udy = \int_{t_1}^{t_2}\left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_{t_1}^{t_2}\left(v\frac{dx}{dt} + u\frac{dy}{dt}\right)dt$$
$$\int_{t_1}^{t_2}(u+iv)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt = \int_{t_1}^{t_2} f(t)\frac{dz}{dt}dt = \int_{t_1}^{t_2} f\big(z(t)\big)z'(t)dt$$

<u>媒介変数を考える＋複素空間をベクトル(実部、虚部)で考える</u>

例1 $\displaystyle\oint_C \frac{dz}{z-a}$ $x = \alpha + R\cos\varphi\,, y = \beta + R\sin\varphi$ $z = a + Re^{i\varphi},\ z' = \frac{dz}{d\varphi} = Rie^{i\varphi}$

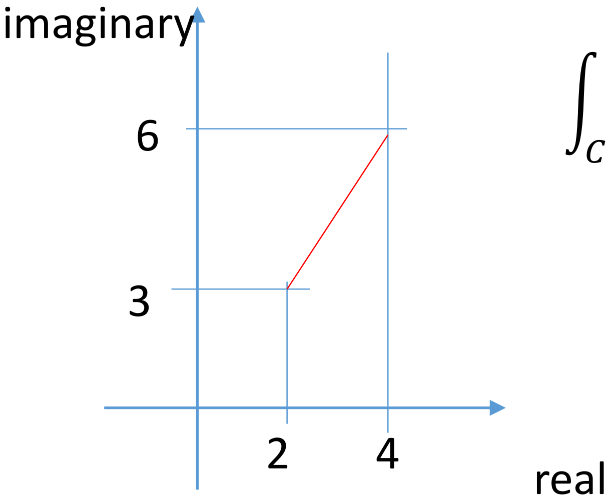
cはこの円周、媒介変数はφ



$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi}\frac{Rie^{i\varphi}}{Re^{i\varphi}}d\varphi = i\int_0^{2\pi}d\varphi = 2\pi i$$

例1’ $$\oint_C (z-a)^n dz = \int_0^{2\pi}\left(Re^{i\varphi}\right)^n Rie^{i\varphi}d\varphi = R^{n+1}i\int_0^{2\pi}e^{(n+1)i\varphi}d\varphi = R^{n+1}i\left.\frac{e^{(n+1)i\varphi}}{(n+1)i}\right|_0^{2\pi} = 0$$

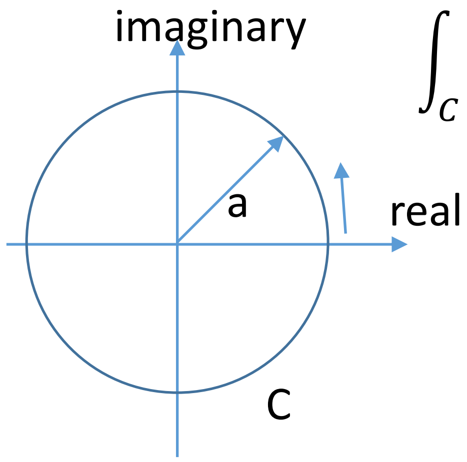
例2　　$f(z) = z^2,$　　積分経路$C: z = 2t + 3ti \ \ (1 \le t \le 2)$



$$\int_C f(z)dz = \int_1^2 (2t+3it)^2 d(2t+3it) = \int_1^2 (2+3i)^2 t^2 \, (2+3i)dt = (2+3i)^3 \int_1^2 t^2 dt$$

$$= (2+3i)^3 \left[\frac{t^3}{3}\right]_1^2 = \frac{7}{3}(2+3i)^3$$

例3



$$\int_C \frac{dz}{z}$$

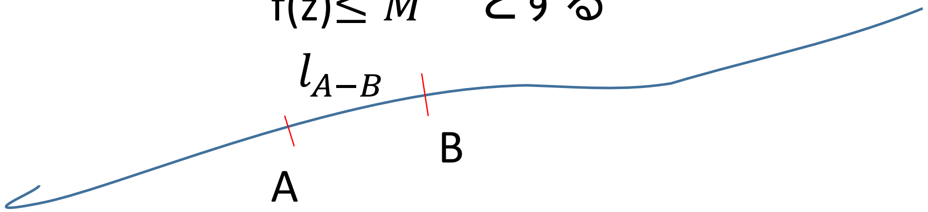
$$z = ae^{it}$$

原点を中心とした半径0の円

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{d(ae^{it})}{ae^{it}} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i$$

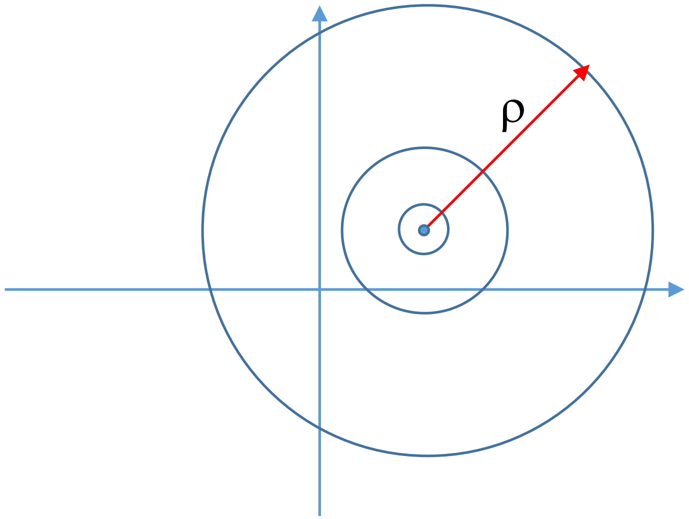
A-B区間で

f(z)≤ M　とする



$$\left|\int_{A-B} f(z)dz\right| \leq M l_{A-B}$$

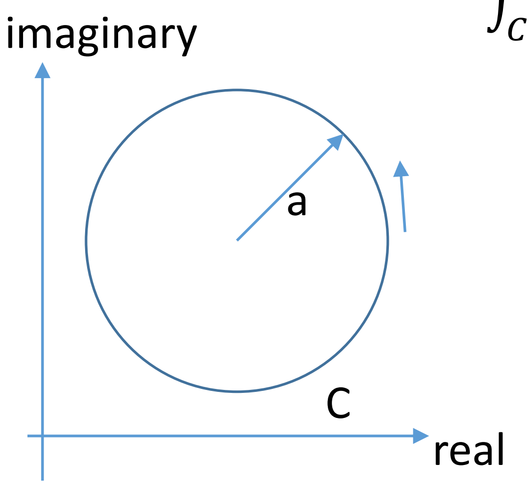
積分の絶対値評価



$$\left|\int_{C} f(z)dz\right| \leq 2\pi\rho M \quad \Rightarrow 0$$

例4　C内で正則な関数f(z)を閉曲線上で積分する

$$\int_C f(z)dz$$



$$f(z) = u(x,y) + v(x,y)i$$

$$\int_C (Mdx + Ndy) = \iint_V \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

グリーンの定理

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

$$= \iint_V \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + i\iint_V \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) dxdy \quad = 0$$

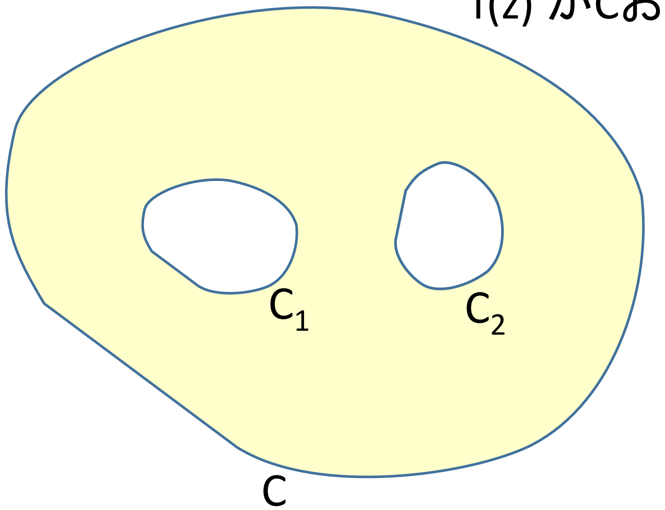
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

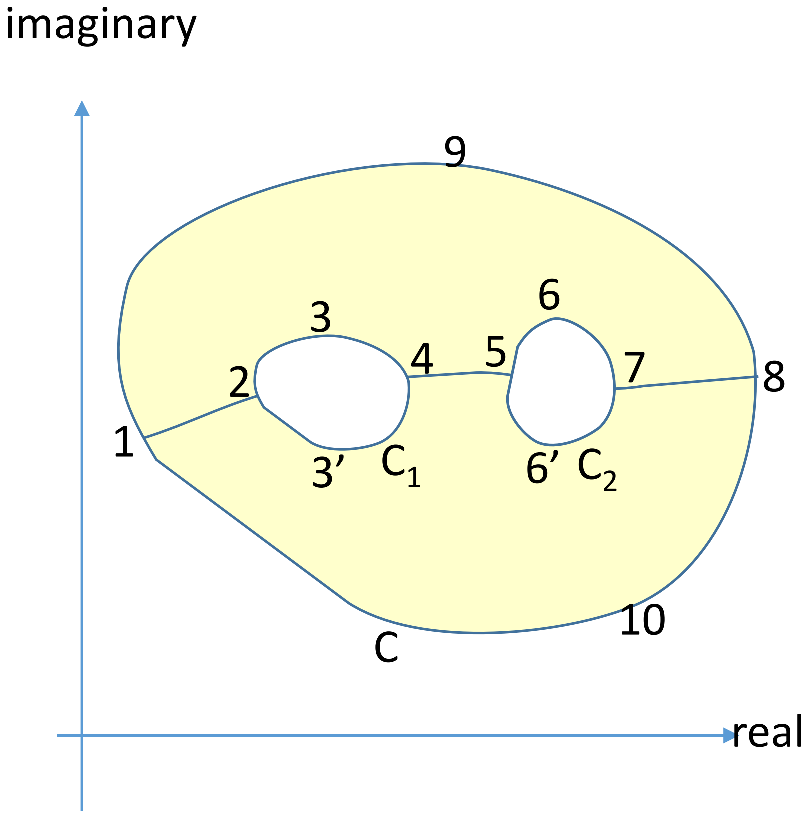
コーシー・リーマンの方程式

コーシーの基本定理
関数f(z)が単連結領域Dにおいて解析的である時は、領域D内にある区分的な滑らかな閉曲線にそっての積分は0に等しい

f(z) がCおよびC1, C2で囲まれた領域と境界線上で解析的（正則）な場合



$$\oint_C f(z)dz = \oint_{c1} f(z)dz + \oint_{c2} f(z)dz$$



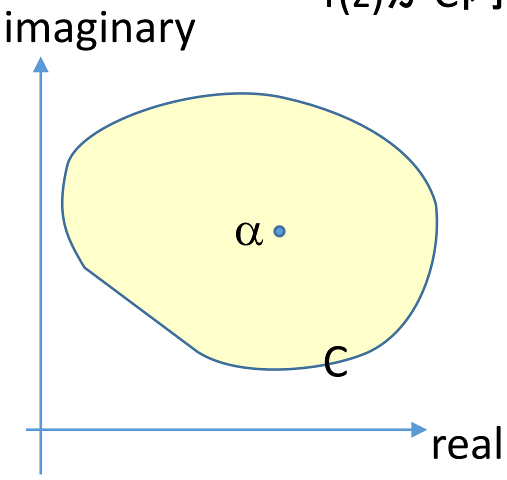
証明　　線で上下に分けたとする。その上と下の閉曲線内では
コーシーの基本定理によって

$$\oint_{1-2-3-4-5-6-7-8-9-1} f(z)dz = \oint_{10-8-7-6'-5-4-3'-2-1-10} f(z)dz = 0$$

$$\oint_{C:10-9} f(z)dz + \oint_{8-7-6'-5-4-3'-2-1} f(z)dz + \oint_{1-2-3-4-5-6-7-8} f(z)dz = 0$$

$$\oint_{C:10-9} f(z)dz = \oint_{3-3\prime} f(z)dz + \oint_{6-6\prime} f(z)dz$$

＝＞境界が複数個ある場合のコーシーの定理

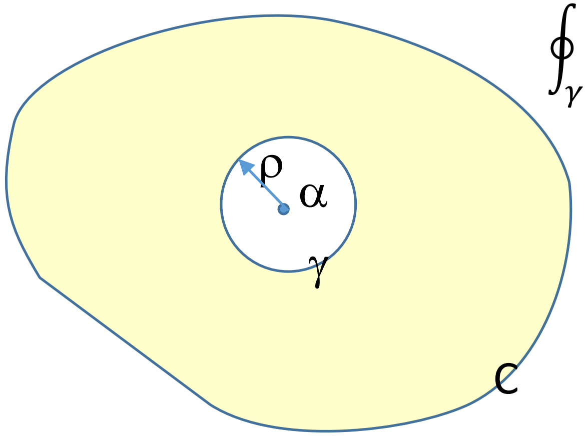


f(z)がC内および境界で解析的であるとき、

$$\int_C \frac{f(z)}{z-\alpha}dz$$

$$\varphi(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$$

を考える



例1より

$$\oint_\gamma \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{Rie^{i\varphi}}{Re^{i\varphi}} d\varphi = i\int_0^{2\pi} d\varphi = 2\pi i$$

↓

$$\int_C \frac{1}{z-\alpha}dz = \oint_\gamma \frac{1}{z-\alpha}dz$$

複数境界のコーシーの定理から

$$\int_C \varphi(\zeta)d\zeta = \oint_\gamma \varphi(\zeta)d\zeta$$

$$|\varphi(\zeta)| < K$$

$$\left|\oint_\gamma \varphi(\zeta)d\zeta\right| \le 2\pi\rho K$$

ρ->0でこの積分は0に　=>

$$\int_C \varphi(\zeta)d\zeta = 0 = \int_C \frac{f(\zeta)-f(z)}{\zeta - z}d\zeta$$

$$\int_C \frac{f(\zeta)}{\zeta - z}d\zeta - f(z)\underbrace{\int_C \frac{1}{\zeta - z}d\zeta}_{2\pi i} = 0$$

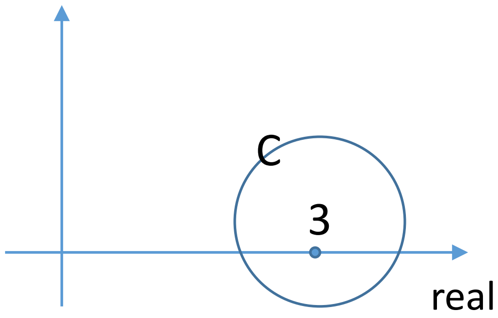
$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

=> コーシーの積分公式

例5

c上での積分 $\int_C \frac{z^2}{z-3}dz$ を求める

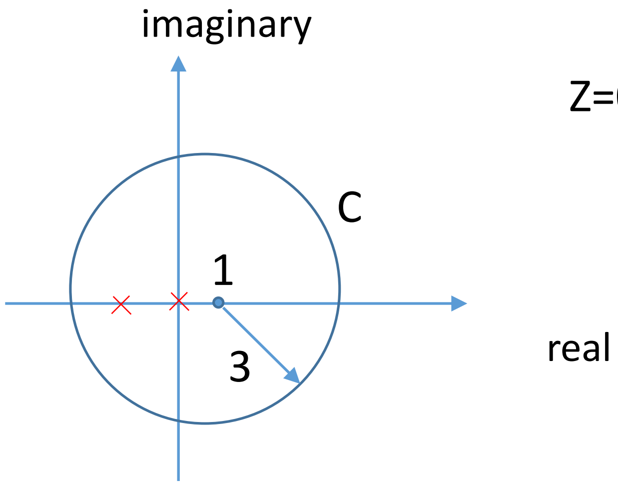
$$f(z) = z^2 \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$



$$f(3) = 3^2 = \frac{1}{2\pi i}\int_C \frac{z^2}{z-3}dz \qquad \int_C \frac{z^2}{z-3}dz = 18\pi i$$

例6 $\int_C \frac{e^z}{z(z+1)}dz$

$$\int_C \frac{e^z}{z(z+1)}dz = \int_C e^z\left(\frac{1}{z} - \frac{1}{z+1}\right)dz = \int_C \frac{e^z}{z}dz - \int_C \frac{e^z}{z+1}dz$$



Z=0, -1 はともにC内

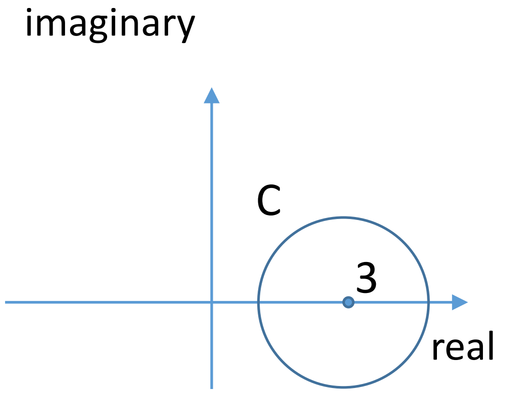
$$\frac{1}{2\pi i}\int_C \frac{e^z}{z}dz = e^0 \qquad \frac{1}{2\pi i}\int_C \frac{e^z}{z+1}dz = e^{-1}$$

$$\int_C \frac{e^z}{z(z+1)}dz = 2\pi i(1 - e^{-1})$$

例7 $\int_C \frac{1}{z^2(z-3)}dz$ $c:\ |z-3|=2$

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

$$f(z)=z^{-2}$$

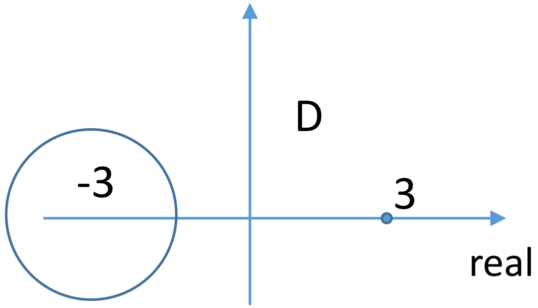


$$f(3)=3^{-2}=\frac{1}{2\pi i}\int_C \frac{z^{-2}}{z-3}dz$$

$$\int_C \frac{1}{z^2(z-3)}dz=\frac{2\pi i}{9}$$

例8 $\int_C \frac{1}{z^2(z-3)}dz$

$D:\ |z+3|=2$????

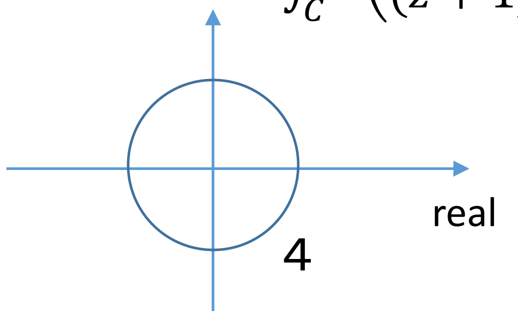


$\frac{1}{z^2(z-3)}$は解析的なので、コーシーの基本定理により0

例7

$$\int_C \left(\frac{1}{(z+1)(z-3)}\right)dz$$

$$\oint_\gamma \frac{dz}{z-\alpha} = 2\pi i$$



$$-\frac{1}{2}\int_C \left(\frac{1}{z+1} - \frac{1}{z-3}\right)dz = -\frac{1}{2}(\int_C \frac{1}{z+1}dz - \int_C \frac{1}{z-3}dz) = -\frac{1}{2}(2\pi i - 2\pi i) = 0$$

$$\int_C \left(\frac{3z-1}{(z+1)(z-3)}\right)dz$$

$$\int_C \left(\frac{1}{z+1} + \frac{2}{z-3}\right)dz = \int_C \frac{1}{z+1}dz + 2\int_C \frac{1}{z-3}dz = 2\pi i + 2\cdot 2\pi i = 6\pi i$$

コーシー型積分

$$\phi(z) = \int_C \frac{\varphi(\zeta)}{(\zeta - z)^k} d\zeta \qquad \frac{\partial}{\partial z}\phi(z) = \int_C \frac{\partial}{\partial z}\frac{\varphi(\zeta)}{(\zeta - z)^k} d\zeta = k\int_C \frac{\varphi(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

$$\phi^{(n)}(z) = \int_C \frac{\partial^n}{\partial z^n}\left(\frac{\varphi(\zeta)}{(\zeta - z)^k}\right) d\zeta = k(k+1)\ldots.(k+n-1)\int_C \frac{\varphi(\zeta)}{(\zeta - z)^{k+n}} d\zeta =$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \Rightarrow \quad f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

コーシー型積分

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$\int_C \frac{1}{z^2(z-3)} dz \qquad D:\ |z| = 2 \qquad f(\zeta) = \frac{1}{\zeta - 3} \qquad \text{n=1, z=0} \qquad f^{(1)}(\zeta) = \frac{-1}{(\zeta-3)^2}$$

$$f^{(1)}(0) = \frac{1!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta)^2} d\zeta = \left.\frac{-1}{(z-3)^2}\right|_{z=0} = -\frac{1}{9}$$

$\zeta \to z$として

$$\int_C \frac{1}{z^2(z-3)} dz = -\frac{2\pi i}{9}$$