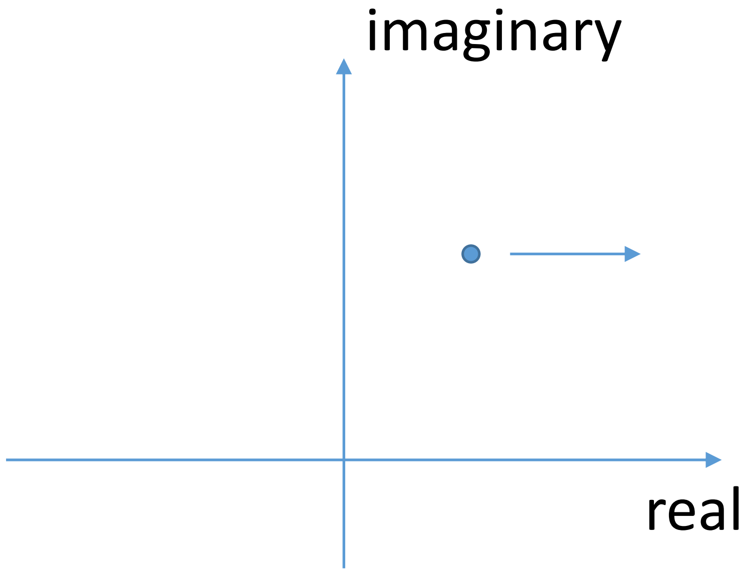
複素関数の実積分への応用II

コーシー・リーマン

複素数では？　$z = x + iy$　$w = f(z) = u(x,y) + iv(x,y)$ 複素数の関数



複素数での増減

$$\Delta w = f(z + \Delta z) - f(z) \qquad \Delta z = \Delta x + i\Delta y$$

実部の増減

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x,y)$$

虚部の増減

$$\Delta v = v(x + \Delta x, y + \Delta y) - v(x,y)$$

1. 変化がRealに沿った場合　$\Delta z = \Delta x$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2. 変化がImaginaryに沿った場合　$\Delta z = \Delta y$

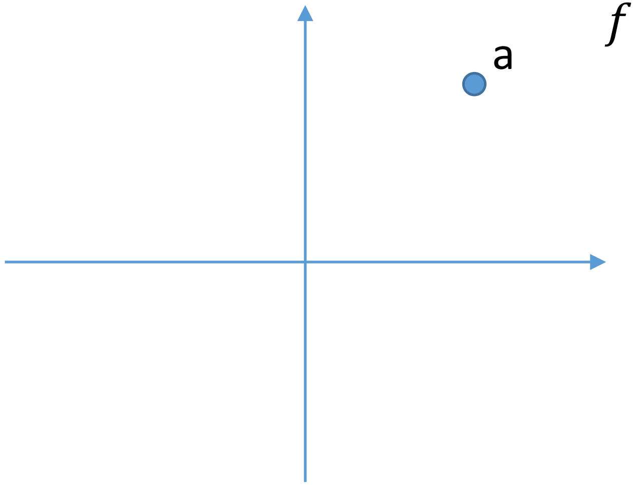
$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y} \rightarrow \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

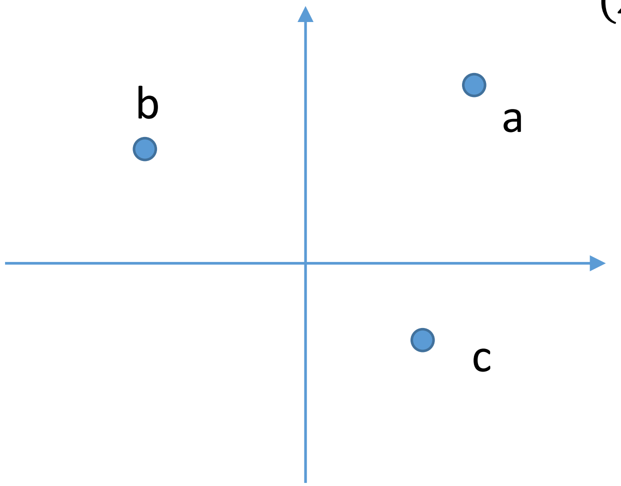
であるならば、微分可能　正則

コーシー・リーマンの条件

孤立特異点

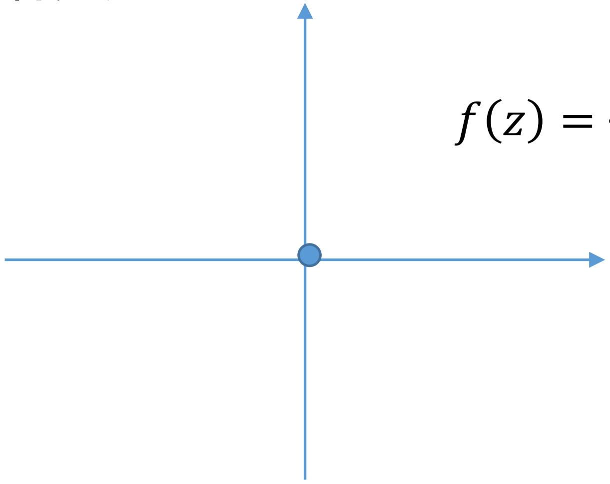


$$f(z) = \frac{1}{z-a}$$



$$f(z) = \frac{1}{(z-a)(z-b)(z-c)}$$

孤立でない特異点



$$f(z) = \frac{1}{\sin(1/z)}$$

$$z = \frac{1}{n\pi}$$

nが∞に近づくとz=0の周りに無限に特異点が生じる

留数の次数の判定

$$f(z) = \frac{1}{\sin z}$$

1つ特異点は$z = 0$

$$\lim_{z\to 0}(z)\frac{1}{\sin z} = \lim_{z\to 0}\frac{1}{\cos z} = 1$$

zの１次をかけて特異点に近づくと有限値をとる
＝＞1次の特異点

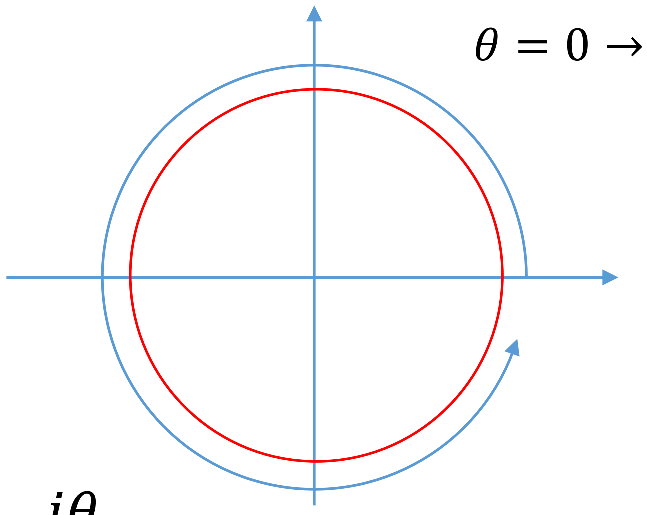
$$\sin z = 0 \quad at\ z = n\pi$$

$$\lim_{z\to n\pi}\frac{(z-n\pi)}{\sin z} = \lim_{z\to n\pi}\frac{1}{\cos z} = (-1)^n$$

z=nπのいずれの特異点も1次

実関数＝＞複素関数を利用して解く

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} \qquad (a > 1)$$



$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta = izd\theta$$

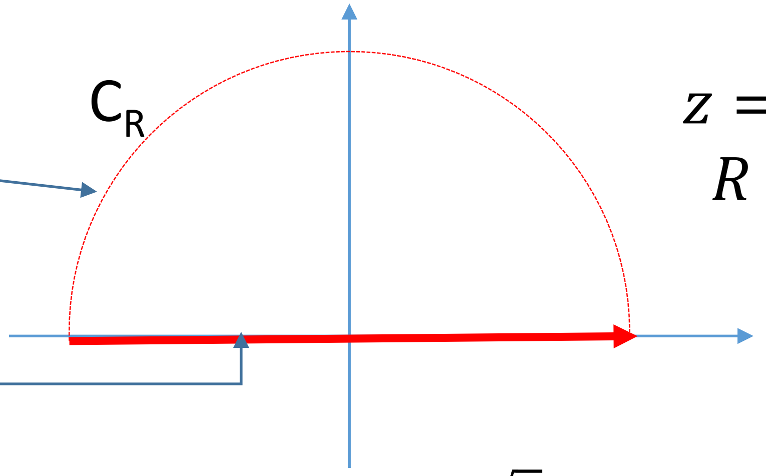
$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \oint \frac{dz/iz}{a + \frac{1}{2}(z + z^{-1})} = \frac{2}{i}\oint \frac{dz}{z^2 + 2az + 1} = \frac{2}{i}\oint \frac{dz}{(z-\alpha)(z-\beta)}$$

単位円の中にあるのはαだけ

$$\alpha, \beta \quad = -a \pm \sqrt{a^2 - 1}$$

$$\frac{2}{i}\oint \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2}{i}\mathrm{Res}\left(\frac{1}{(z-\alpha)(z-\beta)}, \qquad z = \alpha\right) = \frac{2}{i}\frac{2\pi i}{(\alpha - \beta)} = \frac{2\pi}{\sqrt{a^2 - 1}}$$

有利関数の無限大への積分への応用

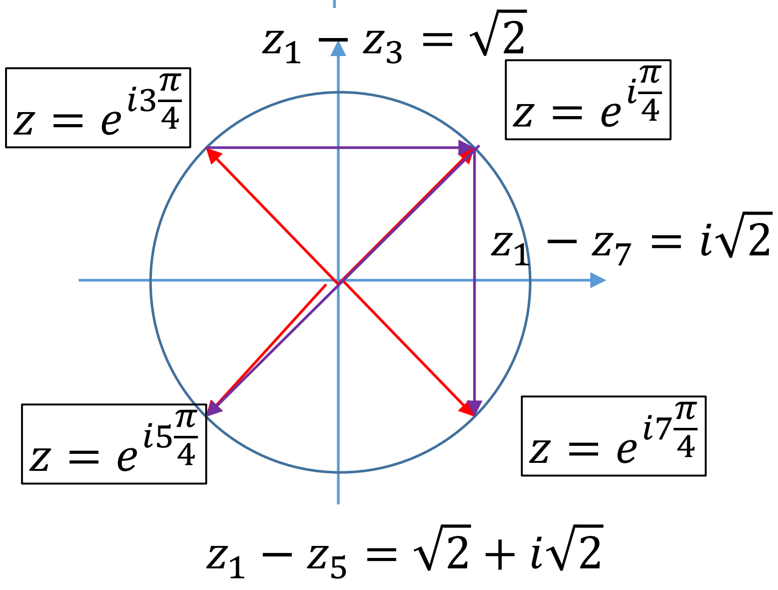


$$z = Re^{i\theta}$$
$$R \to \infty$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$\oint_C \frac{dz}{z^4+1} = \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} \frac{dz}{z^4+1}$$

$z^4 = -1$が特異点＝＞4回回ってz=-1になる。

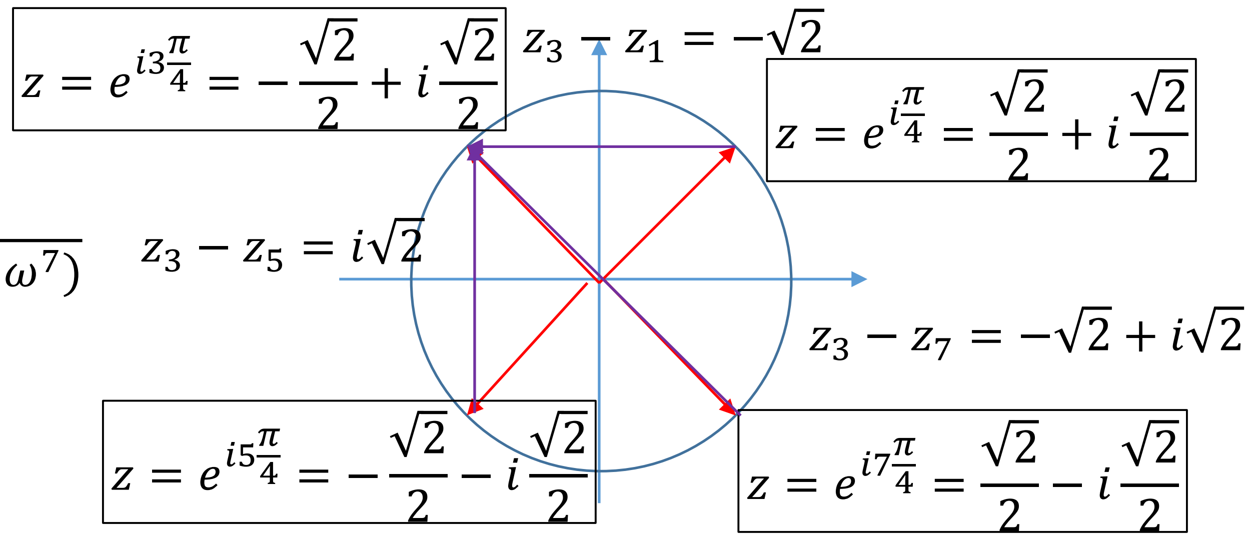


$$\mathrm{Res}\left(\frac{1}{z^4+1}, z=\omega\right) = \lim_{z\to\omega} \frac{(z-\omega)}{(z-\omega)(z-\omega^3)(z-\omega^5)(z-\omega^7)}$$

$$= \frac{1}{\sqrt{2}\left(\sqrt{2}+i\sqrt{2}\right)i\sqrt{2}} = \frac{1}{2\sqrt{2}(i-1)} = \frac{-(1+i)}{4\sqrt{2}} = -\frac{\omega}{4}$$

$$\operatorname{Res}\left(\frac{1}{z^4+1}, z=\omega^3\right) = \lim_{z\to\omega^3}\frac{(z-\omega^3)}{(z-\omega)(z-\omega^3)(z-\omega^5)(z-\omega^7)}$$

$$= \frac{1}{-\sqrt{2}i\sqrt{2}\left(-\sqrt{2}+i\sqrt{2}\right)} = \frac{1}{2\sqrt{2}(1+i)} = \frac{(1-i)}{4\sqrt{2}} = \frac{1}{4\omega}$$



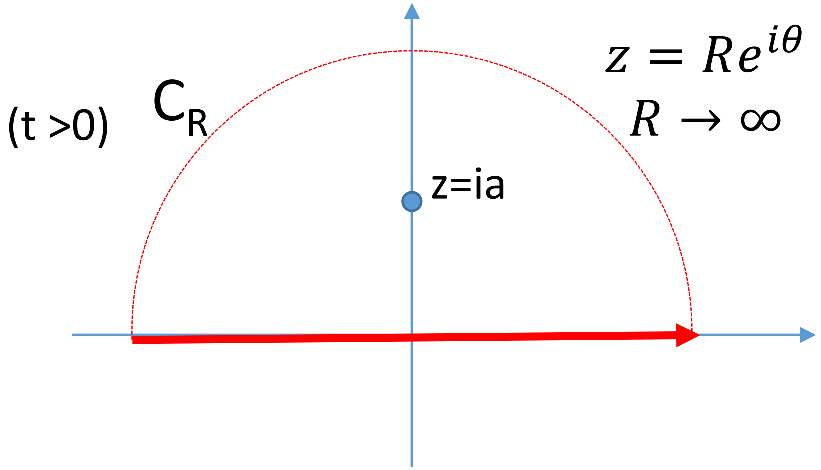
$$\oint \frac{dz}{z^4+1} = 2\pi i\left(\frac{-(1+i)}{4\sqrt{2}} + \frac{(1-i)}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

$$\left|\int_{C_R}\frac{dz}{z^4+1}\right| \le \frac{R}{R^4-1}\int_0^{\pi} d\theta \to 0 \quad for\ R\to\infty$$

$$\int_{-R}^{R}\frac{dx}{x^4+1} = \oint_C \frac{dz}{z^4+1} - \int_{C_R}\frac{dz}{z^4+1} = \frac{\pi}{\sqrt{2}}$$

Fourier-type積分

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx \quad \Rightarrow \quad \int_{-\infty}^{\infty}\frac{e^{itx}}{x^2+a^2}$$



$$\int_C \frac{e^{itz}}{z^2+a^2}dz = \int_C \frac{e^{itz}}{(z-ia)(z+ia)}dz = 2\pi i \mathrm{Res}\left(\frac{e^{itz}}{(z-ia)(z+ia)}, z = ia\right) = 2\pi i\frac{e^{-at}}{2ia} = \frac{\pi}{a}e^{-at}$$

半円の積分は、円弧と弦の足し算なので

$$\int_C \frac{e^{itz}}{z^2+a^2}dz = \int_{-R}^{R}\frac{e^{itx}}{x^2+a^2}dx + \int_{C_R}\frac{e^{itz}}{z^2+a^2}$$

=> 0

$$\int_{-\infty}^{\infty}\frac{e^{itx}}{x^2+a^2}dx = \frac{\pi}{a}e^{-at}$$

$$\int_{-\infty}^{\infty}\frac{e^{itx}}{x^2+a^2}dx = \int_{-\infty}^{\infty}\frac{\cos tx}{x^2+a^2}dx + i\int_{-\infty}^{\infty}\frac{\sin tx}{x^2+a^2}dx = \frac{\pi}{a}e^{-at}$$

t=1 では

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx = \frac{\pi}{a}e^{-a}$$

Fourier-type積分　その２

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-at}$$ をd/dtして

$$\int_{-\infty}^{\infty} \frac{x\sin tx}{x^2+a^2}\,dx = \frac{\pi}{a}(-a)e^{-at} = -ae^{-at}$$

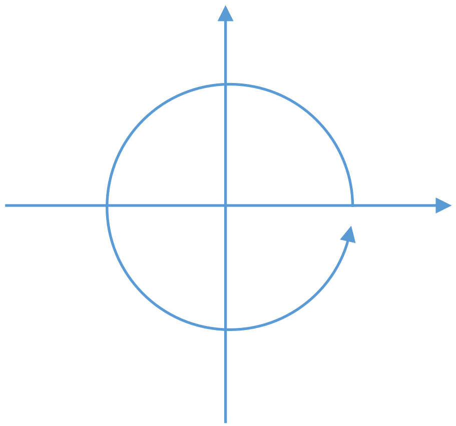
複素関数の実関数積分：積分領域を工夫する。特異点周り＝＞留数定理

分岐点

w=f(z)で1つのzに対して複数のwが存在する＝＞多価関数　＝＞分岐点

例：　$w = z^{1/2}$　　$z = re^{i\theta}$

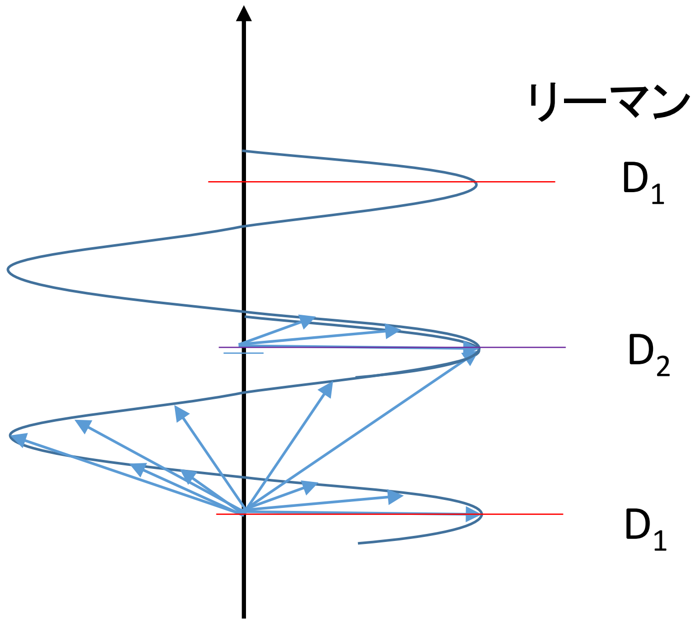
$$w = \sqrt{r}e^{i\theta/2}$$



θ	0	π	2π	3π	4π
z	$\sqrt{r}$	-$\sqrt{r}$	$\sqrt{r}$	-$\sqrt{r}$	$\sqrt{r}$
w	$\sqrt{r}$	i$\sqrt{r}$	-$\sqrt{r}$	-i$\sqrt{r}$	$\sqrt{r}$

2価関数

リーマン面



例 : $w = \log z$

$$z = re^{i\theta}$$

$$w = \log r + i(\theta + 2n\pi) \qquad n = \cdots. -2, -1, 0, 1, 2, \dots.$$

無限多価関数　＝＞無数のリーマン面

例 : $w = z^{1/n}$

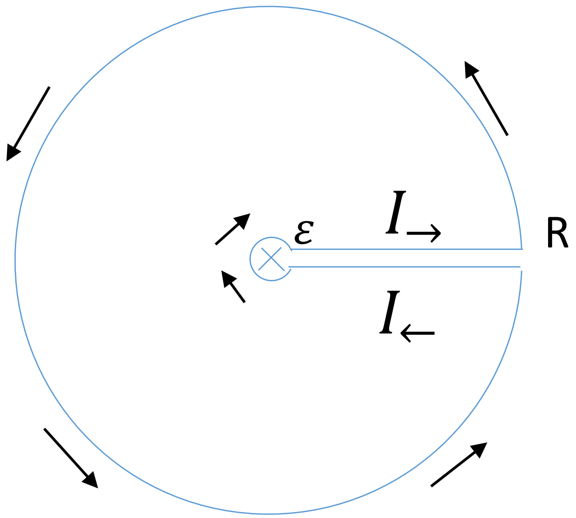
$$z = re^{i\theta}$$

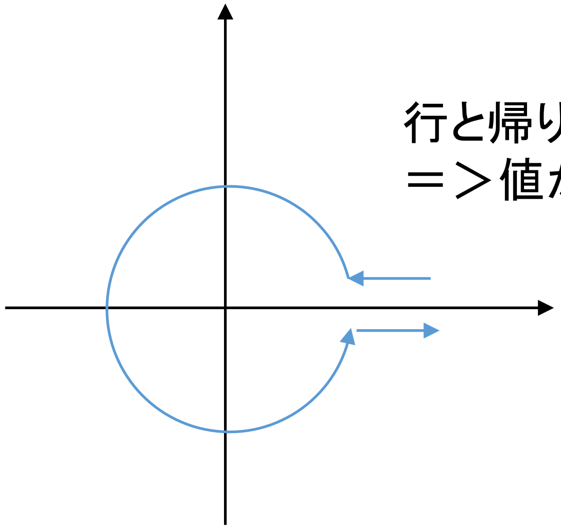
$$w = z^{1/n} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\log r + i(\theta + 2n\pi))} \qquad n = \cdots. -2, -1, 0, 1, 2, \dots.$$

無数のリーマン面

複数のリーマン面だと積分が少し困ることに。

例：特異点を迂回させた場合





行と帰りで異なるリーマン面
＝＞値が異なる。

例：

$$I = \int_0^{\infty} z^{b-1} \frac{1}{z+1} dz$$

0<b<1　ε, R として、0と∞に漸近させる。

$$I_{\rightarrow} = \int_{\epsilon}^{R} z^{b-1} \frac{1}{z+1} dz = \int_{\epsilon}^{R} x^{b-1} \frac{1}{x+1} dx$$

$$I_{\leftarrow} = \int_{R}^{\varepsilon} z^{b-1} \frac{1}{z+1} dz = \int_{R}^{\varepsilon} (xe^{2\pi i})^{b-1} \frac{1}{x+1} dx = -e^{2\pi ib} \int_{\varepsilon}^{R} x^{b-1} \frac{1}{x+1} dx = -e^{2\pi ib} I_{\rightarrow}$$

$$I_{外円} = \int_0^{2\pi} \frac{z^{b-1}}{z+1} dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{b-1}}{Re^{i\theta}+1} iRe^{i\theta} d\theta = \int_0^{2\pi} \frac{iR^b e^{ib\theta}}{Re^{i\theta}+1} d\theta = \int_0^{2\pi} \frac{ibR^{b-1}e^{ib\theta}}{e^{i\theta}} d\theta = 0$$

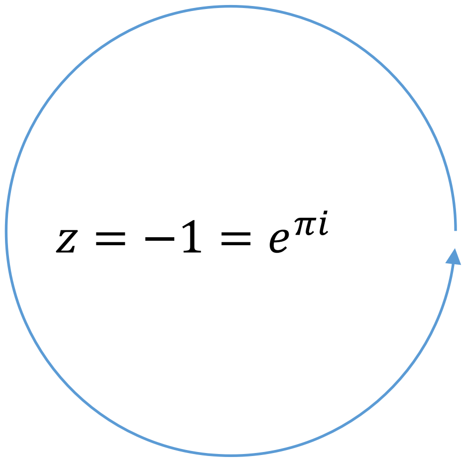
b<1

$R \to \infty$

$$I_{小円} = \int_{2\pi}^{0} \frac{z^{b-1}}{z+1} dz = \int_{2\pi}^{0} \frac{(\varepsilon e^{i\theta})^{b-1}}{\varepsilon e^{i\theta}+1} i\varepsilon e^{i\theta} d\theta = i\int_{2\pi}^{0} \frac{(\varepsilon e^{i\theta})^{b}}{\varepsilon e^{i\theta}+1} id\theta = 0$$

b>0

$\varepsilon \to 0$

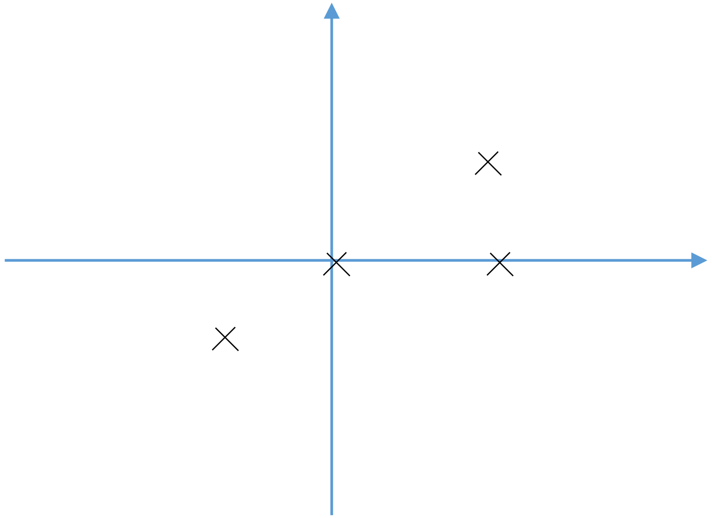


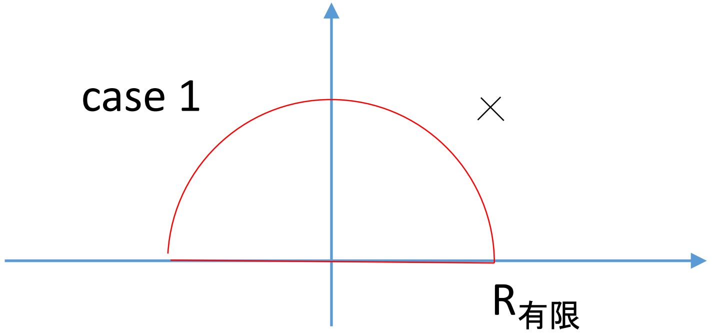
$$Res(-1) = (e^{\pi i})^{b-1} = -e^{\pi i b}$$

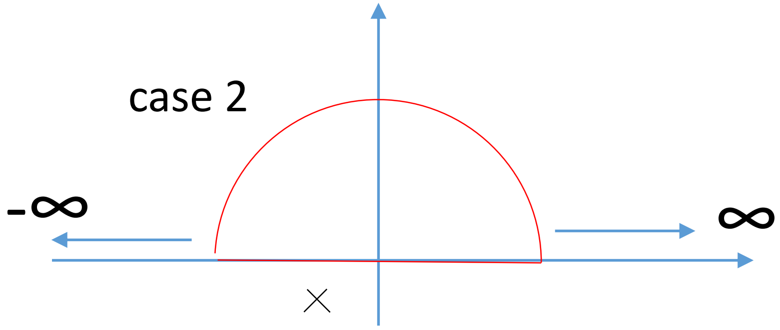
$$I = \frac{-2\pi i e^{\pi b i}}{1 - e^{2\pi b i}} = \frac{-2\pi i}{e^{-\pi b i} - e^{\pi b i}} = \frac{\pi}{\sin \pi b}$$

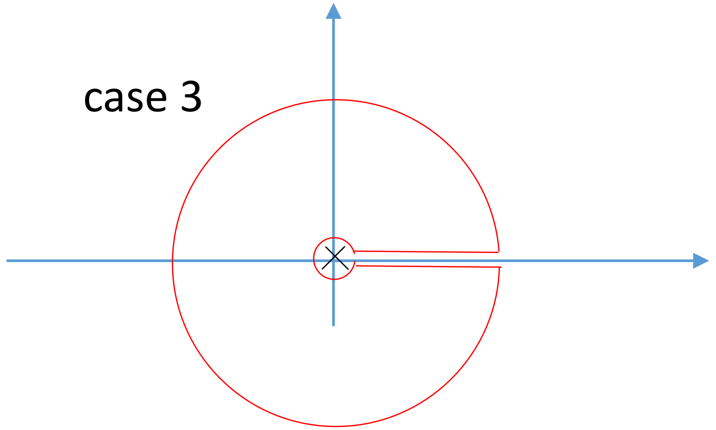
実積分への複素積分

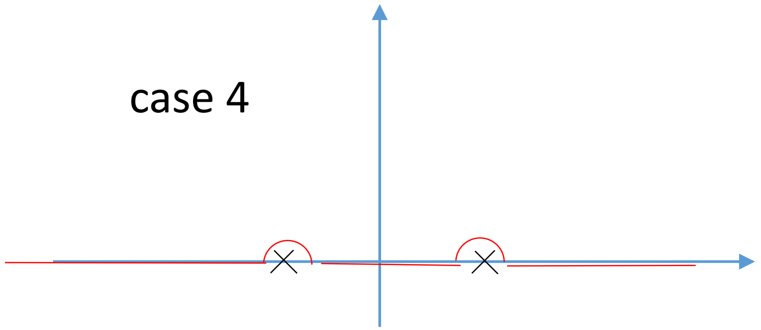
積分範囲のとり方：　特異点を避ける。







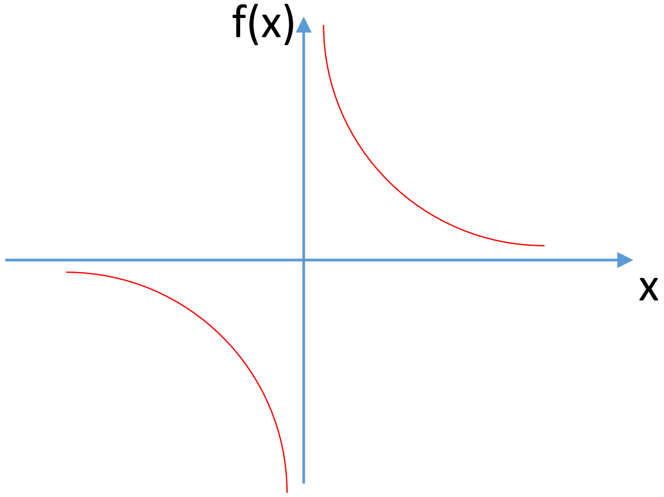


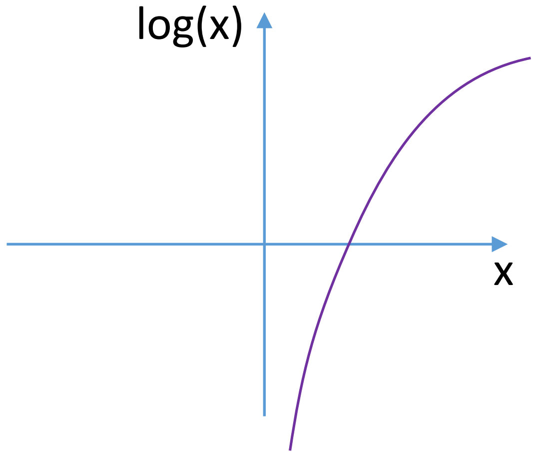


$$f(x) = \frac{1}{x}$$

$$\int f(x)dx = [\log x]$$

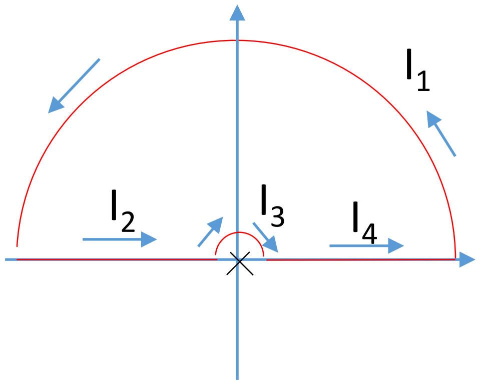
$$\int_0^R f(x)dx = [\log x]_0^R = \log R - \log 0 \to \infty$$





$$\int_{-R}^R f(x)dx = [\log x]_{-R}^{R} \qquad ????$$

$$f(z) = \frac{1}{z}$$



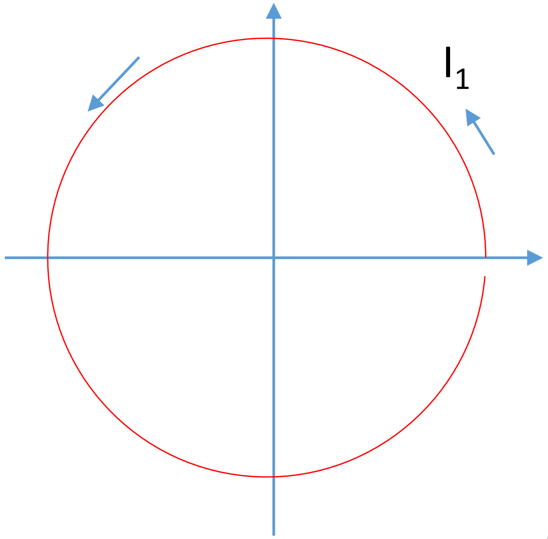
$I = I_1 + I_2 + I_3 + I_4 = 0$

$$I_1 = \int_{CR} \frac{dz}{z} = \int_0^\pi \frac{d(Re^{i\theta})}{Re^{i\theta}} = i[\theta]_0^\pi \to i\pi$$

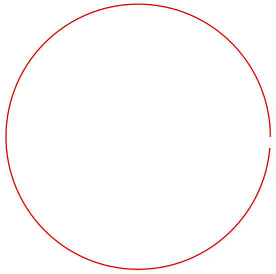
$$I_2 = \int_{-R}^{\varepsilon} \frac{1}{x}dx \qquad I_4 = \int_{\varepsilon}^{R} \frac{1}{x}dx$$

$$I_3 = \int_{C\varepsilon} \frac{dz}{z} = \int_\pi^0 \frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = -i[\theta]_0^\pi \to -i\pi$$

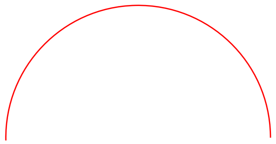
$$f(z) = \frac{1}{z}$$



$$I_1 = \int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{d(Re^{i\theta})}{Re^{i\theta}} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

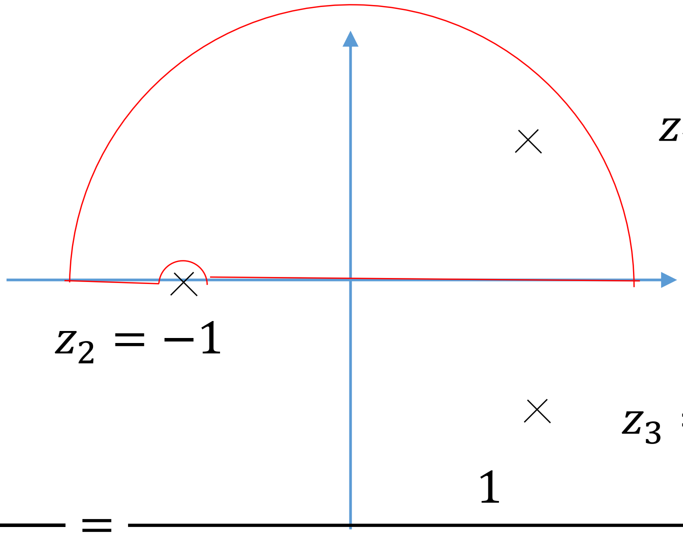


$= 2\pi i$



$= \pi i$

$$f(x) = \frac{1}{1+x^3} \qquad \int_{-\infty}^{\infty} f(x)dx$$



$$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2 = -1$$

$$z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$Res(f(z), z_1) = \lim_{z \to z_1} \frac{(z-z_1)}{(z-z_1)(z-z_2)(z-z_3)} = \frac{1}{(\frac{1}{2}+\frac{\sqrt{3}}{2}i+1)(\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = \frac{1}{(\frac{3}{2}+\frac{\sqrt{3}}{2}i)\sqrt{3}i}$$

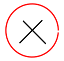
$$= \frac{2}{3\sqrt{3}i-3} = \frac{2}{3}\frac{\sqrt{3}i+1}{-4} = \frac{-1}{6}(1+\sqrt{3}i)$$

$$\int_C \frac{1}{z^3+1}dz = 2\pi i \frac{-1}{6}\left(1+\sqrt{3}i\right) = \frac{2\sqrt{3}}{3}\pi - \frac{\pi}{3}i$$

$$= \int_{\pi}^{0} \frac{dz}{(z-z_1)(z-z_2)(z-z_3)} = \int_{\pi}^{0} \frac{\frac{dz}{d\theta}d\theta}{e^{i\theta}(-1-\frac{1}{2}-\frac{\sqrt{3}}{2}i+\varepsilon e^{i\theta})(-1-\frac{1}{2}+\frac{\sqrt{3}}{2}i+\varepsilon e^{i\theta})}$$

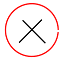
$$= \int_{\pi}^{0} i \frac{d\theta}{(-\frac{3}{2}-\frac{\sqrt{3}}{2}i)(-\frac{3}{2}+\frac{\sqrt{3}}{2}i)} = i\frac{1}{\frac{9}{4}+\frac{3}{4}}\int d\theta = \frac{i}{3}\pi$$

$$= \int_{\pi}^{0} i \frac{d\theta}{(-\frac{3}{2} - \frac{\sqrt{3}}{2} i)(-\frac{3}{2} + \frac{\sqrt{3}}{2} i)} = i \frac{1}{\frac{9}{4} + \frac{3}{4}} \int d\theta = \frac{i}{3} \pi$$



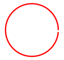
$$Res(f(z), z_2) = \lim_{z \to z_2} \frac{(z - z_2)}{(z - z_1)(z - z_2)(z - z_3)} = \frac{1}{(-1 - \frac{1}{2} - \frac{\sqrt{3}}{2} i)(-1 - \frac{1}{2} + \frac{\sqrt{3}}{2} i)} = \frac{1}{(-\frac{3}{2} - \frac{\sqrt{3}}{2} i)(-\frac{3}{2} + \frac{\sqrt{3}}{2} i)}$$

$$= \frac{1}{\frac{9}{4} + \frac{3}{4}} = \frac{1}{3}$$



$$= \int_{C\varepsilon} \frac{dz}{z^3 + 1} = 2\pi i \frac{1}{3}$$

 =



$$\frac{\quad}{2}$$

$$= \frac{\pi}{3} i$$

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} \qquad z = re^{i\theta} \qquad \cos\theta = \frac{z + \frac{1}{z}}{2} \qquad dz = ie^{i\theta}d\theta = izd\theta$$

$$I = \oint_{|z|=1} \frac{dz}{iz} \frac{1}{5 - 2(z + 1/z)} = i\oint_{|z|=1} \frac{dz}{2z^2 - 5z + 2} = \frac{i}{2}\oint_{|z|=1} \frac{dz}{(z - \frac{1}{2})(z - 2)} = \frac{2\pi}{3}$$

留数を使って

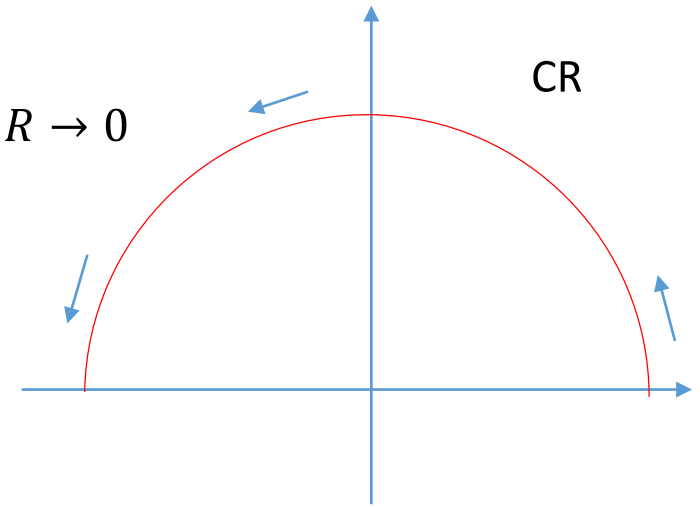
半径1の円に入るのはz=1/2だけ

ジョルダンの補題

$$I_R = \int_{CB} e^{iaz} f(z) dz \ \to 0 \qquad R \to \infty$$

$f(z) \ \to 0 \; for \;\; R \to 0$

$a > 0$



$$|I_R| = \left| \int_{CB} e^{iaz} f(z) dz \right| \le \int_{CR} |e^{iaz}| |f(z)| |dz|$$

$$z = Re^{i\theta}$$

$$dz = iRe^{i\theta} d\theta, \; |dz| = R d\theta, \; \left|e^{iaz}\right| = e^{-aR\sin\theta}$$

$$|I_R| \le \int_0^{\pi} e^{-aR\sin\theta} \left| f(Re^{i\theta}) \right| R d\theta \qquad |f(z)| < \varepsilon$$

$$|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\theta} d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta} d\theta \le 2\varepsilon R \int_0^{\pi/2} e^{-(2aR/\pi)\theta} d\theta = \frac{\varepsilon\pi}{a}(1 - e^{-aR}) \le \frac{\pi}{a}\varepsilon$$

$$\sin\theta \ge \frac{2\theta}{\pi} \quad 0 \le \theta \le \frac{\pi}{2}$$