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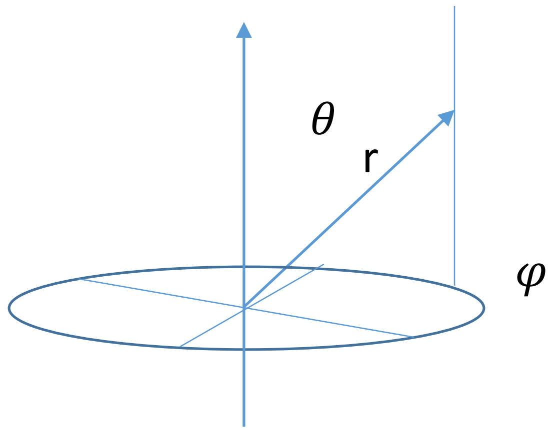
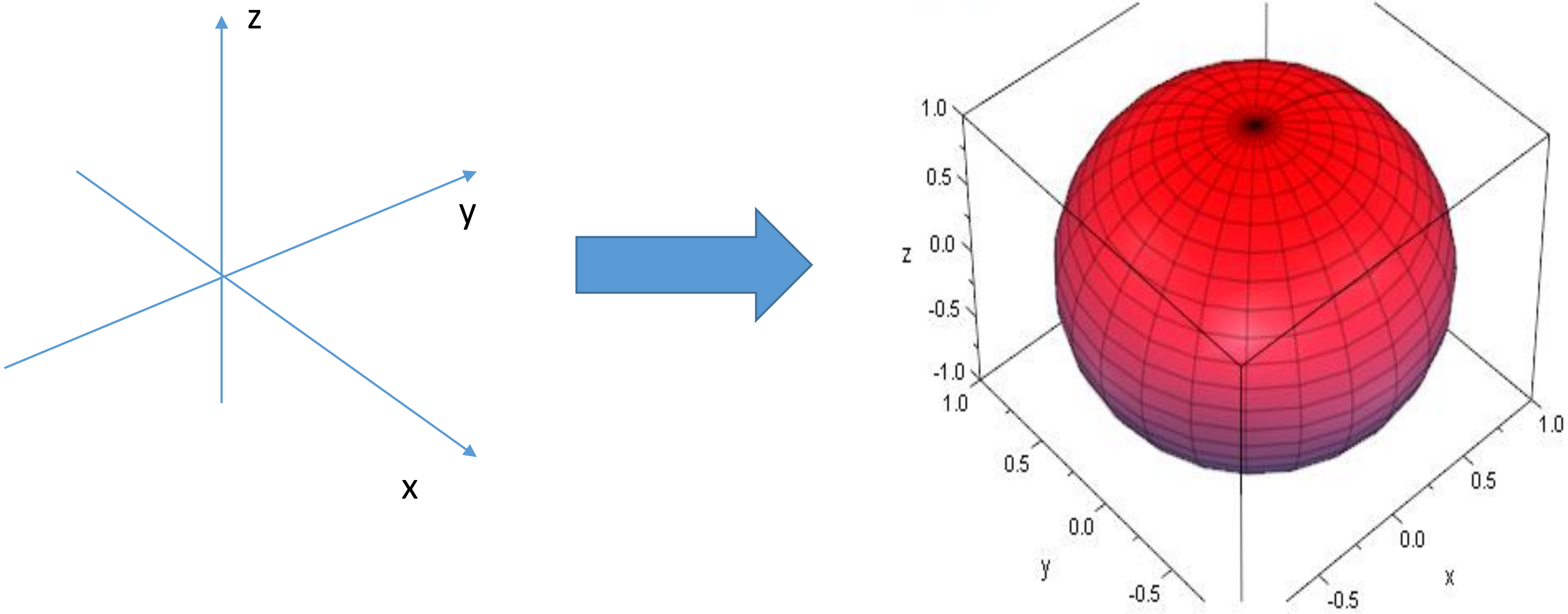
応用数学　A 第3回

曲線座標におけるベクトルの演算



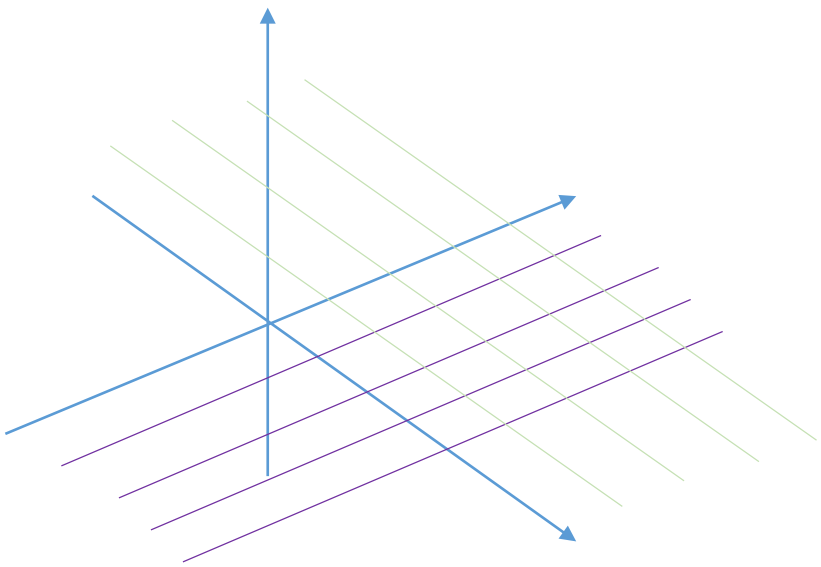
$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

例えば、x(r,u,v), y(r,u,v), z(r,u,v)として　$x = r\cos u \sin v,$　$y = r \sin u \sin v\,,$　$z = r\cos v$



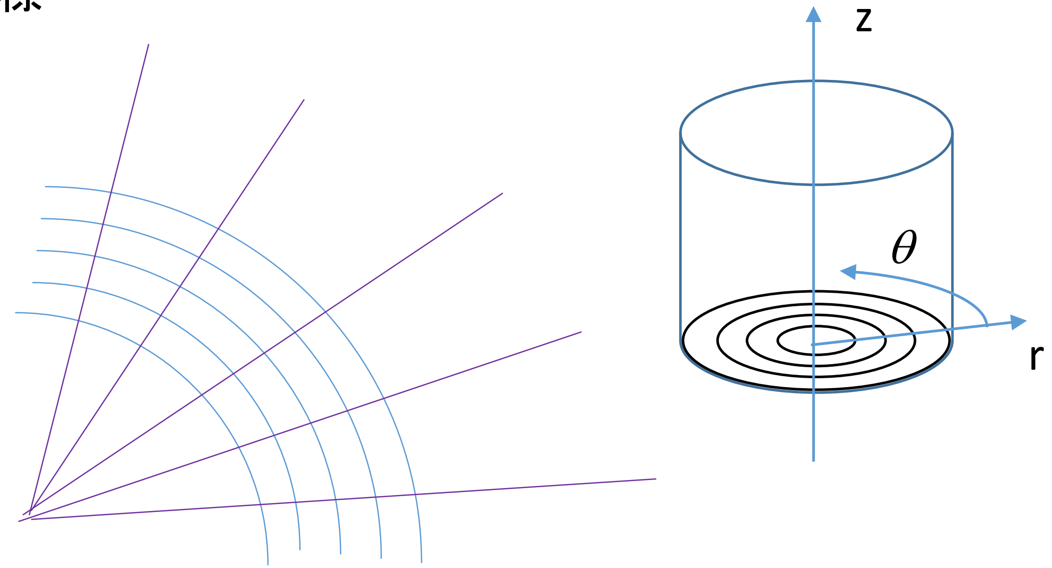
直交座標系

それぞれの軸の交差点は直角
＝＞1つの軸のことは他の軸ではできない

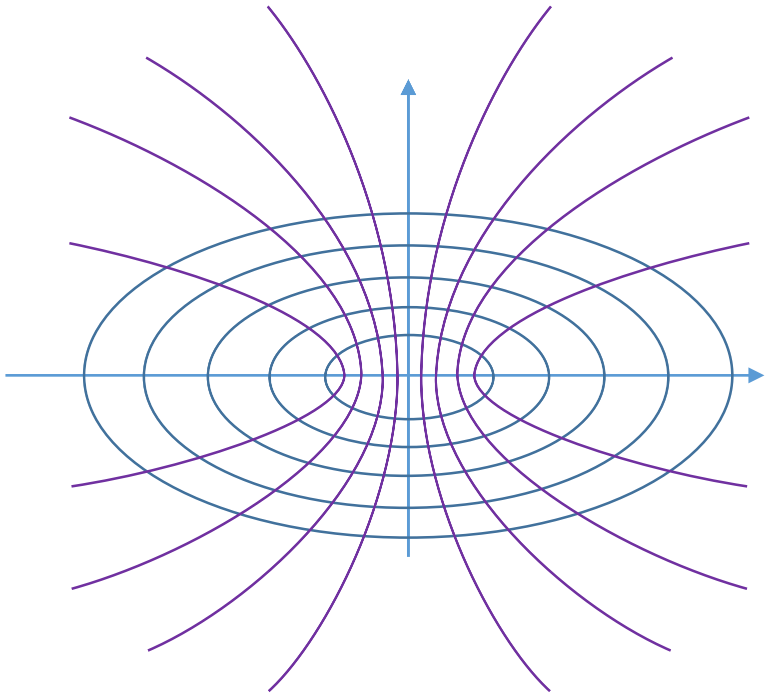


直交曲線座標

円筒座標

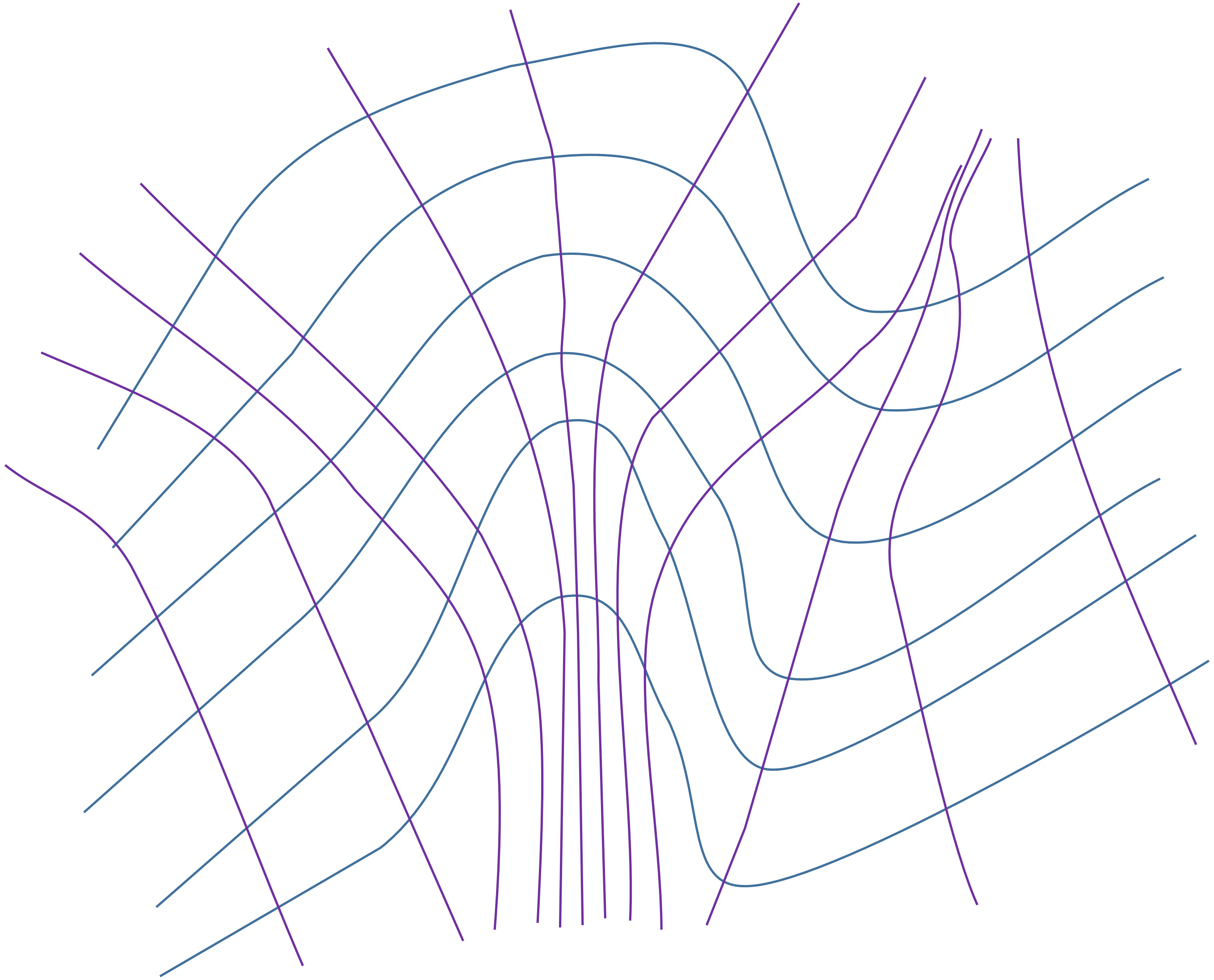


楕円柱座標



直交座標系

自分なりの直交座標系を作ってみよう



直交系

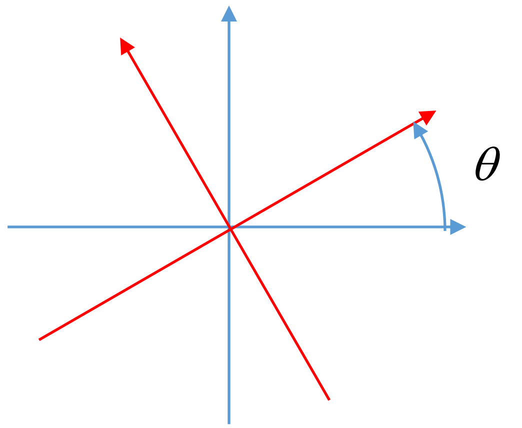
$$\vec{A}\cdot\vec{B} = 0$$

ベクトルで直交といえば

x軸　　y軸

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$$

$$\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}\cdot\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = 0$$



直交行列

$$\begin{matrix} \vec{A} & \vec{B} & \vec{C} \end{matrix}$$

$$\left(\begin{pmatrix} \square \\ \square \\ \square \end{pmatrix} \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix} \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix} \right)$$

$$\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

$$\vec{A} \cdot \vec{B} = 0 \qquad |\vec{A}| = 1$$

$$\vec{B} \cdot \vec{C} = 0 \qquad |\vec{B}| = 1$$

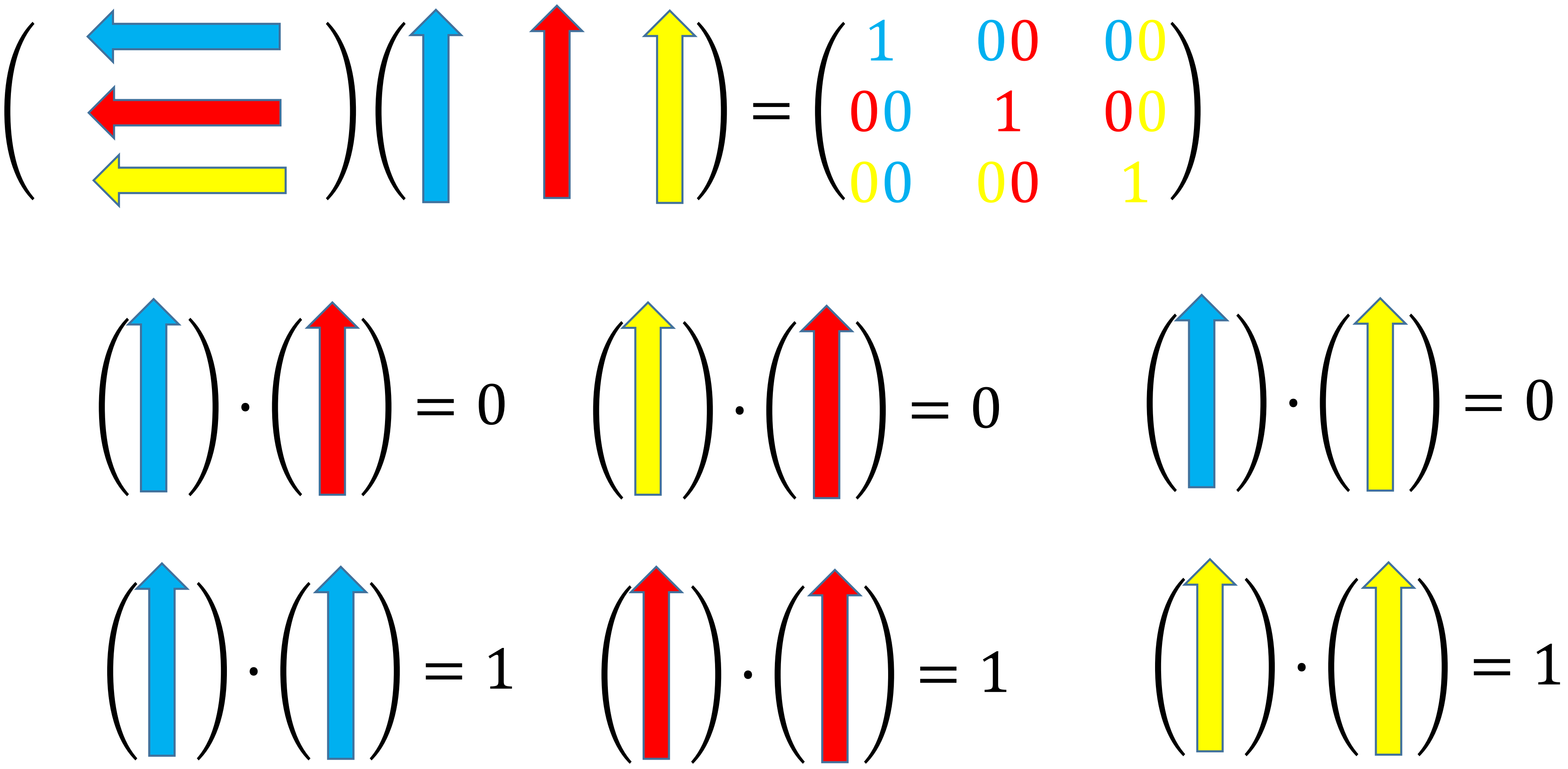
$$\vec{C} \cdot \vec{A} = 0 \qquad |\vec{C}| = 1$$

$$P^t = P^{-1} \qquad P\vec{a} \cdot P\vec{b} = \vec{a} \cdot \vec{b} \qquad |P\vec{a}| = |\vec{a}|$$

転置行列　　逆行列

証明

$$P^t = P^{-1} \qquad P^tP = \tilde{1}$$



証明

$$P\vec{a}\cdot P\vec{b} = \vec{a}\cdot\vec{b}$$

ベクトルの内積　　　　　　　　　　　　　　行列の掛け算

$$\vec{a}\cdot\vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 = (a_1 \quad a_2 \quad a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = {}^t\vec{a}\vec{b}$$

$$P\vec{a}\cdot P\vec{b} = {}^t(P\vec{a})(P\vec{b}) = {}^t\vec{a}\,{}^tP(P\vec{b}) = {}^t\vec{a}({}^tPP\vec{b}) = {}^t\vec{a}\vec{b} = \vec{a}\cdot\vec{b}$$

$${}^t(AB) = {}^tB\,{}^tA$$

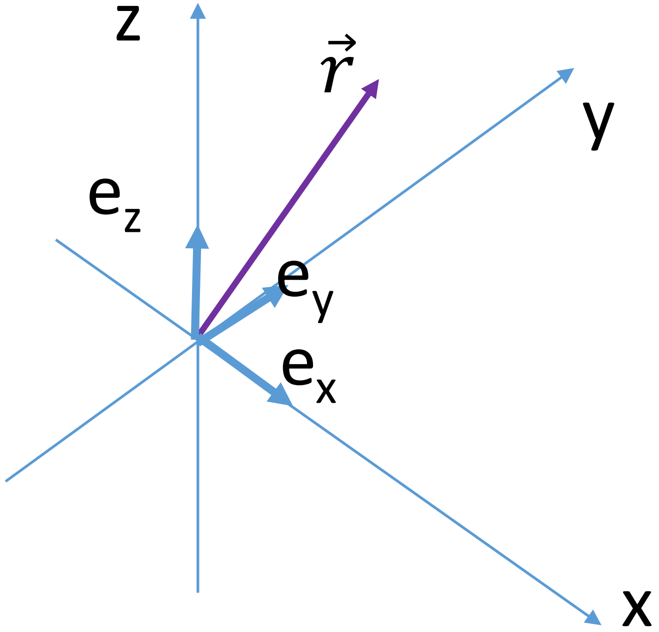
$$|P\vec{a}| = |\vec{a}| \qquad P\vec{a}\cdot P\vec{b} = \vec{a}\cdot\vec{b} \quad ➡ \quad P\vec{a}\cdot P\vec{a} = \vec{a}\cdot\vec{a}$$

$$\left|\vec{A}\right|^2 = \vec{A}\cdot\vec{A}$$

行列は、座標変換の形もしている。

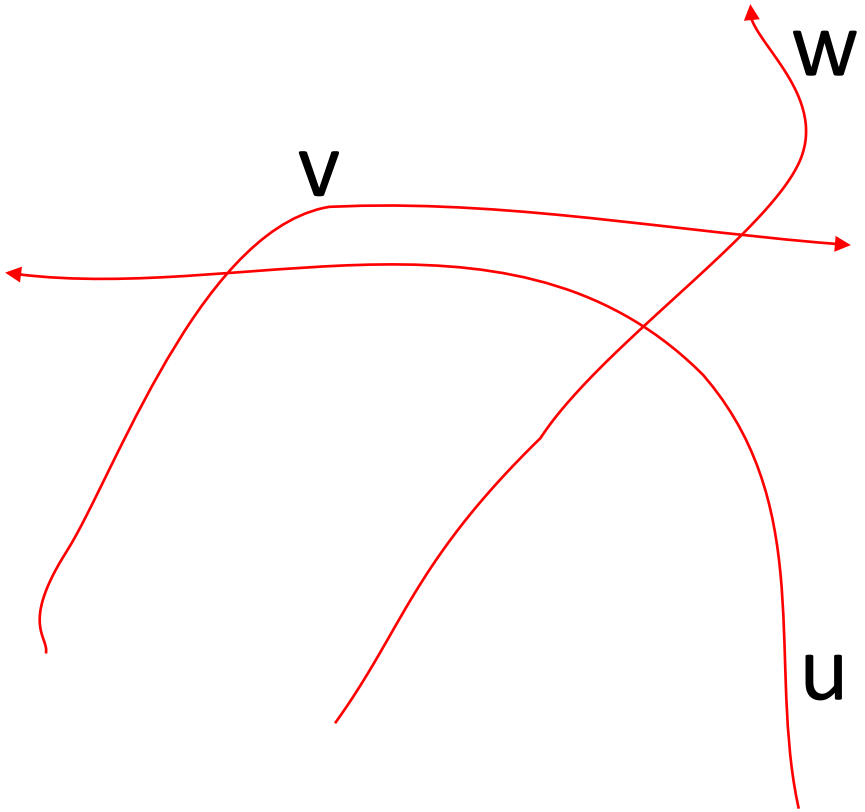
$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ r \\ z \end{pmatrix}$$

直交系の座標変換



$$u = u(x, y, z)$$
$$v = v(x, y, z)$$
$$w = w(x, y, z)$$

$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$



座標変換 $(x,y,z) \Rightarrow (u,v,w)$

ラメの係数

$$H_i = \left|\frac{\partial \vec{r}}{\partial u_i}\right|, \qquad \frac{\partial \vec{r}}{\partial u_i} = H_i \vec{e_i}, \qquad \vec{e_i} \cdot \vec{e_j} = \delta_{ij}, \qquad \vec{e_i} \times \vec{e_j} = \varepsilon_{ijk}\vec{e_k}$$

直交座標の条件

$grad\ u_i$を求める

$$u = u(x,y,z) = u(x(u,v,w), y(u,v,w), z(u,v,w))$$

まず $$\frac{du}{du} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial u}$$

$$grad\ u_i = \begin{pmatrix} \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ \frac{\partial u_i}{\partial z} \end{pmatrix} \qquad \frac{\partial \vec{r}}{\partial u_i} = \begin{pmatrix} \frac{\partial x}{\partial u_i} \\ \frac{\partial y}{\partial u_i} \\ \frac{\partial z}{\partial u_i} \end{pmatrix}$$ なので

$$grad\ u_i \cdot \frac{\partial \vec{r}}{\partial u_i} = \frac{\partial u_i}{\partial x}\frac{\partial x}{\partial u_i} + \frac{\partial u_i}{\partial y}\frac{\partial y}{\partial u_i} + \frac{\partial u_i}{\partial z}\frac{\partial z}{\partial u_i} = 1$$

$$grad\ u_i \cdot \frac{\partial \vec{r}}{\partial u_i} = grad\ u_i \cdot H_i \vec{e_i} = 1$$

したがって $$grad\ u_i = k\vec{e_i} = \frac{1}{H_i}\vec{e_i}$$

$grad\ u_i$を求める

$$H_i = \left|\frac{\partial \vec{r}}{\partial u_i}\right|, \qquad \frac{\partial \vec{r}}{\partial u_i} = H_i \vec{e_i}, \qquad \vec{e_i} \cdot \vec{e_j} = \delta_{ij}, \qquad \vec{e_i} \times \vec{e_j} = \varepsilon_{ijk} \vec{e_k}$$

$$grad\ u_i = k\vec{e_i} = \frac{1}{H_i}\vec{e_i}$$

$grad\ \phi$ を求める

$$grad\ \phi(u,v,w) = \frac{\partial \phi}{\partial u} grad\ u + \frac{\partial \phi}{\partial v} grad\ v + \frac{\partial \phi}{\partial w} grad\ w$$

$$grad\ u = \frac{1}{H_u}\vec{e_u}, \qquad grad\ v = \frac{1}{H_v}\vec{e_v}, \qquad grad\ w = \frac{1}{H_w}\vec{e_w},$$

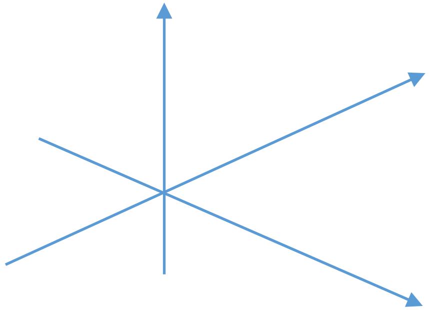
$$grad\ \phi = \frac{1}{H_u}\frac{\partial \phi}{\partial u}\vec{e_u} + \frac{1}{H_v}\frac{\partial \phi}{\partial v}\vec{e_v} + \frac{1}{H_w}\frac{\partial \phi}{\partial w}\vec{e_w}$$

ここまでは一般形

円筒座標での$grad\ \phi$

一般形 $grad\ \phi = \frac{1}{H_u}\frac{\partial \phi}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial \phi}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial \phi}{\partial w}\overrightarrow{e_w}$

$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$



$\vec{r} = x\cdot\overrightarrow{e_x} + y\cdot\overrightarrow{e_y} + z\cdot\overrightarrow{e_z}$ と定義すれば、

$$\overrightarrow{e_r} = \frac{1}{H_r}\frac{\partial \vec{r}}{\partial r} \qquad \overrightarrow{e_\varphi} = \frac{1}{H_\varphi}\frac{\partial \vec{r}}{\partial \varphi} \qquad \overrightarrow{e_z} = \frac{1}{H_z}\frac{\partial \vec{r}}{\partial z}$$

ここで、 $H_r = \left|\frac{\partial \vec{r}}{\partial r}\right|, \ H_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right|, \ H_z = \left|\frac{\partial \vec{r}}{\partial z}\right|$

$$\frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

$$H_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos\varphi^2 + \sin\varphi^2} = 1$$

$$H_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \left|\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}\right| = r\sqrt{\cos\varphi^2 + \sin\varphi^2} = r$$

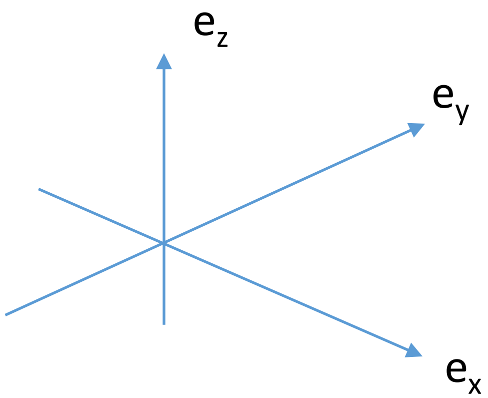
$$H_z = 1$$

答えは $$grad\ \phi = \frac{\partial \phi}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial \phi}{\partial \varphi}\overrightarrow{e_\varphi} + \frac{\partial \phi}{\partial z}\overrightarrow{e_z}$$

極座標での$grad\ \phi$

一般形 $grad\ \phi = \frac{1}{H_u}\frac{\partial\phi}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial\phi}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial\phi}{\partial w}\overrightarrow{e_w}$

$$x = r\cos\theta\cos\varphi,\qquad y = r\cos\theta\sin\varphi,\qquad z = r\sin\theta$$



ここで、 $H_r = \left|\frac{\partial\vec{r}}{\partial r}\right|,\ H_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right|,\ H_\theta = \left|\frac{\partial\vec{r}}{\partial\theta}\right|$

$$\frac{\partial\vec{r}}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{pmatrix}$$

$$\frac{\partial\vec{r}}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ r\sin\theta \end{pmatrix} = r\begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$H_r = \left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{(\cos\varphi^2 + \sin\varphi^2)\cos\theta^2 + \sin\theta^2} = 1$$

$$H_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \left|\begin{pmatrix} -r\cos\theta\sin\varphi \\ r\cos\theta\cos\varphi \\ 0 \end{pmatrix}\right| = r\cos\theta\sqrt{\cos\varphi^2 + \sin\varphi^2}$$

$$= r\cos\theta$$

$$H_\theta = r$$

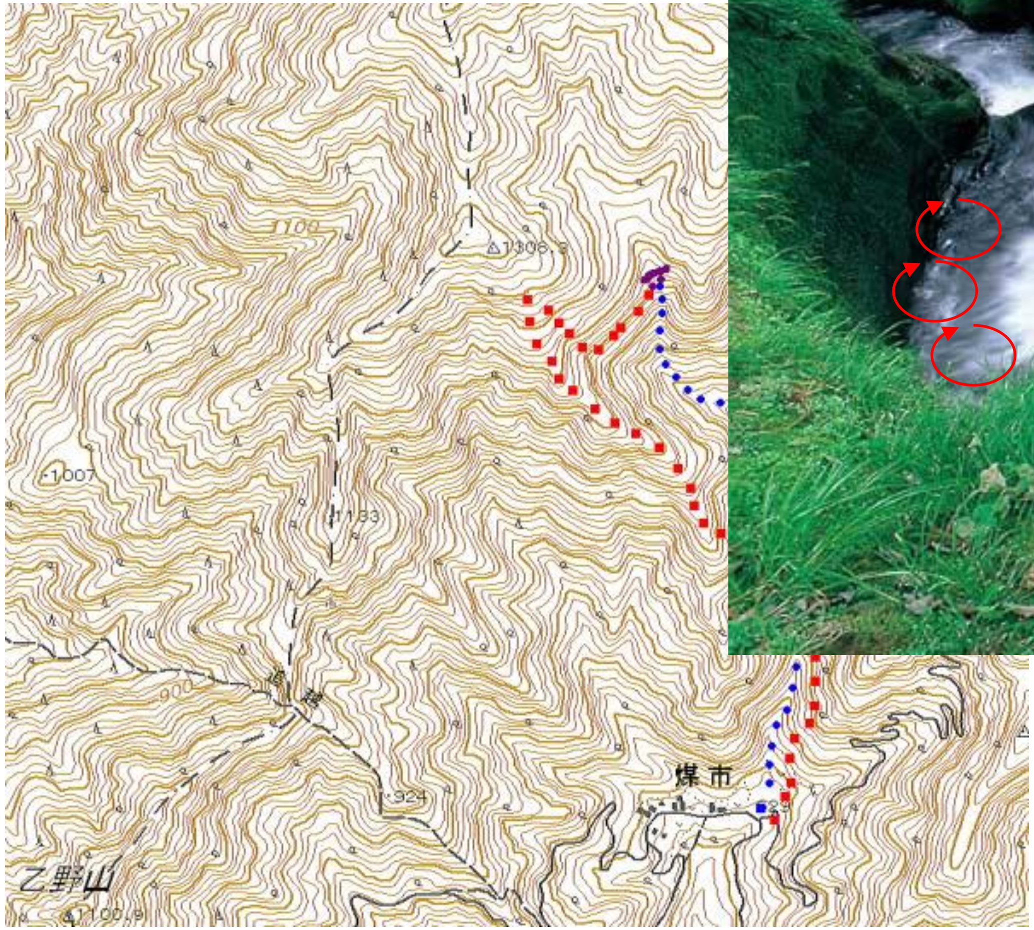
答えは $grad\ \phi = \frac{\partial\phi}{\partial r}\overrightarrow{e_r} + \frac{1}{r\cos\theta}\frac{\partial\phi}{\partial\varphi}\overrightarrow{e_\varphi} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\overrightarrow{e_\theta}$

ベクトル演算

gradient $\nabla \phi$

divergence $\nabla \cdot \phi$

rotation $\nabla \times \phi$



発散(divergence)　証明

$$div\ \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad \Rightarrow \quad div\ \vec{a} = \frac{\frac{\partial}{\partial u}(a_u H_v H_w) + \frac{\partial}{\partial v}(a_v H_w H_u) + \frac{\partial}{\partial w}(a_w H_u H_v)}{H_u H_v H_w}$$

$$\frac{\partial \vec{r}}{\partial u} = H_u \overrightarrow{e_u} = \frac{\partial x}{\partial u}\overrightarrow{e_x} + \frac{\partial y}{\partial u}\overrightarrow{e_y} + \frac{\partial z}{\partial u}\overrightarrow{e_z}$$

$$\frac{\partial \vec{r}}{\partial v} = H_v \overrightarrow{e_v} = \frac{\partial x}{\partial v}\overrightarrow{e_x} + \frac{\partial y}{\partial v}\overrightarrow{e_y} + \frac{\partial z}{\partial v}\overrightarrow{e_z}$$

$$\frac{\partial \vec{r}}{\partial w} = H_w \overrightarrow{e_w} = \frac{\partial x}{\partial w}\overrightarrow{e_x} + \frac{\partial y}{\partial w}\overrightarrow{e_y} + \frac{\partial z}{\partial w}\overrightarrow{e_z}$$

⇔ 直交化だから

$$\overrightarrow{e_x} = \frac{1}{H_u}\frac{\partial x}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial x}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial x}{\partial w}\overrightarrow{e_w}$$

$$\overrightarrow{e_y} = \frac{1}{H_u}\frac{\partial y}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial y}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial y}{\partial w}\overrightarrow{e_w}$$

$$\overrightarrow{e_z} = \frac{1}{H_u}\frac{\partial z}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial z}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial z}{\partial w}\overrightarrow{e_w}$$

$$\nabla = \frac{\partial}{\partial x}\overrightarrow{e_x} + \frac{\partial}{\partial y}\overrightarrow{e_y} + \frac{\partial}{\partial z}\overrightarrow{e_z} = \frac{1}{H_u}\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right)\overrightarrow{e_u} + \cdots = \frac{1}{H_u}\frac{\partial}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial}{\partial w}\overrightarrow{e_w}$$

$$\vec{a} = a_u\overrightarrow{e_u} + a_v\overrightarrow{e_v} + +a_w\overrightarrow{e_w}$$

$$\nabla \cdot \vec{a} = \nabla \cdot (a_u\overrightarrow{e_u} + a_v\overrightarrow{e_v} + +a_w\overrightarrow{e_w}) = (\nabla a_u)\cdot\overrightarrow{e_u} + a_u(\nabla \cdot \overrightarrow{e_u}) \ldots = \frac{1}{H_u}\frac{\partial a_u}{\partial u} + \frac{1}{H_u H_v H_w}\frac{\partial H_v H_w}{\partial u} + \cdots = \frac{1}{H_u H_v H_w}\frac{\partial a_u H_v H_w}{\partial u} + \cdots$$

分配法則

部分微分

発散（divergence）

一般形 $$div\ \vec{a} = \frac{\frac{\partial}{\partial u}(a_u H_v H_w) + \frac{\partial}{\partial v}(a_v H_w H_u) + \frac{\partial}{\partial w}(a_w H_u H_v)}{H_u H_v H_w}$$

円筒座標

$$H_r = \left|\frac{\partial \vec{r}}{\partial r}\right|, \quad H_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right|, \quad H_z = \left|\frac{\partial \vec{r}}{\partial z}\right|$$

$$H_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos\varphi^2 + \sin\varphi^2} = 1$$

$$H_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \left|\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}\right| = r\sqrt{\cos\varphi^2 + \sin\varphi^2} = r$$

$$H_z = 1$$

$$H_r, H_\varphi, H_z = 1, r, 1$$

$$div\ \vec{a} = \frac{a_r}{r} + \frac{\partial a_r}{\partial r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

$$\frac{\frac{\partial}{\partial u}(a_u H_v H_w)}{H_u H_v H_w}$$

発散（divergence）

一般形 $$div\ \vec{a} = \frac{\frac{\partial}{\partial u}(a_u H_v H_w) + \frac{\partial}{\partial v}(a_v H_w H_u) + \frac{\partial}{\partial w}(a_w H_u H_v)}{H_u H_v H_w}$$

極座標

$$x = r\cos\theta\cos\varphi, \qquad y = r\cos\theta\sin\varphi, \qquad z = r\sin\theta$$

$$H_r = \left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{(\cos\varphi^2 + \sin\varphi^2)\cos\theta^2 + \sin\theta^2} = 1$$

$$H_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \left|\begin{pmatrix} -r\cos\theta\sin\varphi \\ r\cos\theta\cos\varphi \\ 0 \end{pmatrix}\right| = r\cos\theta\sqrt{\cos\varphi^2 + \sin\varphi^2} = r\cos\theta$$

$$H_\theta = r$$

$$H_r, H_\varphi, H_\theta = 1, r\cos\theta, r$$

$$div\ \vec{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial a_\varphi}{\partial\varphi} + \frac{1}{r\cos\theta}\frac{\partial(a_\theta\cos\theta)}{\partial\theta}$$

回転（Rotation）

$$rot\ \vec{a} = \overrightarrow{e_u}\frac{\frac{\partial(a_w H_w)}{\partial v} - \frac{\partial(a_v H_v)}{\partial w}}{H_v H_w} + \overrightarrow{e_v}\frac{\frac{\partial(a_u H_u)}{\partial w} - \frac{\partial(a_w H_w)}{\partial u}}{H_w H_u} + \overrightarrow{e_w}\frac{\frac{\partial(a_v H_v)}{\partial u} - \frac{\partial(a_u H_u)}{\partial v}}{H_u H_v}$$

円筒座標　$H_r, H_\varphi, H_z = 1, r, 1$

$$rot\ \vec{a} = \overrightarrow{e_r}\left(\frac{1}{r}\frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z}\right) + \overrightarrow{e_\varphi}\left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right) + \overrightarrow{e_z}\left(\frac{a_\varphi}{r} + \frac{\partial a_\varphi}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right)$$

回転（Rotation）

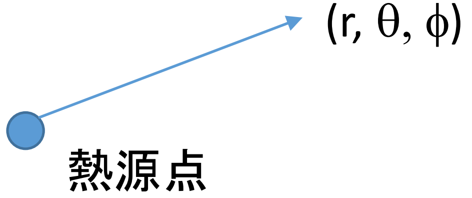
極座標

$$rot\ \vec{a} = \overrightarrow{e_u}\frac{\frac{\partial(a_w H_w)}{\partial v} - \frac{\partial(a_v H_v)}{\partial w}}{H_v H_w} + \overrightarrow{e_v}\frac{\frac{\partial(a_u H_u)}{\partial w} - \frac{\partial(a_w H_w)}{\partial u}}{H_w H_u} + \overrightarrow{e_w}\frac{\frac{\partial(a_v H_v)}{\partial u} - \frac{\partial(a_u H_u)}{\partial v}}{H_u H_v}$$

$$H_r, H_\varphi, H_\theta = 1, r\cos\theta, r$$

$$rot\ \vec{a}$$
$$= \overrightarrow{e_r}\left(\frac{1}{r\cos\theta}\frac{\partial a_\theta}{\partial\varphi} - \frac{1}{r\cos\theta}\frac{\partial a_\varphi \cos\theta}{\partial\theta}\right) + \overrightarrow{e_\varphi}\left(\frac{1}{r}\frac{\partial a_r}{\partial\theta} - \frac{1}{r}\frac{\partial r a_\theta}{\partial r}\right) + \overrightarrow{e_\theta}\left(\frac{1}{r}\frac{\partial r a_\varphi}{\partial r} - \frac{1}{r\cos\theta}\frac{\partial a_r}{\partial\varphi}\right)$$

例題　　極座標での熱伝導方程式



$$\frac{\partial U}{\partial t} = \kappa \nabla^2 U = \kappa \nabla \cdot \nabla U$$

$$\nabla \phi = \frac{1}{H_u}\frac{\partial \phi}{\partial u}\overrightarrow{e_u} + \frac{1}{H_v}\frac{\partial \phi}{\partial v}\overrightarrow{e_v} + \frac{1}{H_w}\frac{\partial \phi}{\partial w}\overrightarrow{e_w}$$

$$H_r, H_\varphi, H_\theta = 1, r\cos\theta, r$$

極座標

$$\nabla \cdot \vec{a} = \frac{\frac{\partial}{\partial u}(a_u H_v H_w) + \frac{\partial}{\partial v}(a_v H_w H_u) + \frac{\partial}{\partial w}(a_w H_u H_v)}{H_u H_v H_w}$$

$$\nabla \cdot \nabla U = \frac{\frac{\partial}{\partial u}\left(H_v H_w \frac{1}{H_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(H_w H_u \frac{1}{H_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(H_u H_v \frac{1}{H_w}\frac{\partial U}{\partial w}\right)}{H_u H_v H_w}$$

$$= \frac{\frac{\partial}{\partial r}\left(r\cos\theta \cdot r \frac{1}{1}\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(r \cdot 1 \frac{1}{r\cos\theta}\frac{\partial U}{\partial \varphi}\right) + \frac{\partial}{\partial \theta}\left(1 \cdot r\cos\theta \frac{1}{r}\frac{\partial U}{\partial \theta}\right)}{1 \cdot r\cos\theta \cdot r}$$

$$= \frac{\frac{\partial}{\partial r}\left(r^2\cos\theta \frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\cos\theta}\frac{\partial U}{\partial \varphi}\right) + \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial U}{\partial w}\right)}{r^2\cos\theta}$$

もし、θ、φ依存性がなければ、

$$\frac{\partial U}{\partial t} = \frac{\frac{\partial}{\partial r}\left(r^2 \cos\theta \frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\cos\theta}\frac{\partial U}{\partial \varphi}\right) + \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial U}{\partial \theta}\right)}{r^2 \cos\theta}$$

$$= \frac{\frac{\partial}{\partial r}\left(r^2 \cos\theta \frac{\partial U}{\partial r}\right)}{r^2 \cos\theta} \quad = \frac{\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)}{r^2}$$

$$= \frac{1}{r^2}\left(2r\frac{\partial U}{\partial r} + r^2 \frac{\partial^2 U}{\partial r^2}\right) = \frac{2}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2}$$

もし$f = rU$としたら、 $\frac{1}{r}\frac{\partial f}{\partial t} = \frac{\partial U}{\partial t}$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial}{\partial r}(rU) = \frac{\partial}{\partial r}\left(U + r\frac{\partial U}{\partial r}\right) = \frac{\partial U}{\partial r} + \frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2} = 2\frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2}$$

$$\frac{1}{r}\frac{\partial f}{\partial t} = \frac{1}{r}\frac{\partial^2 f}{\partial r^2} \Rightarrow \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial r^2}$$

直交曲線座標におけるベクトルの演算

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

極座標 $x = r\cos\varphi\sin\theta,\quad y = r\sin\varphi\sin\theta\,,\quad z = r\cos\theta$

円筒座標 $x = r\cos\varphi,\quad y = r\sin\varphi,\quad z = z$

楕円柱座標 $x = a\cosh u\ \cos v\,,\quad y = a\sinh u\sin v\quad z = z$

$$H_i = \left|\frac{\partial\vec{r}}{\partial u_i}\right|,\qquad \frac{\partial\vec{r}}{\partial u_i} = H_i\vec{e_i},\qquad \vec{e_i}\cdot\vec{e_j} = \delta_{ij},\qquad \vec{e_i}\times\vec{e_j} = \varepsilon_{ijk}\vec{e_k}$$