応用数学　A

積分定理

グリーンの定理

Gaussの定理　$\int divBdV = \int B \cdot ndS$

Stokesの定理　$\int \nabla \times E \cdot bdS = \int E \cdot dr$

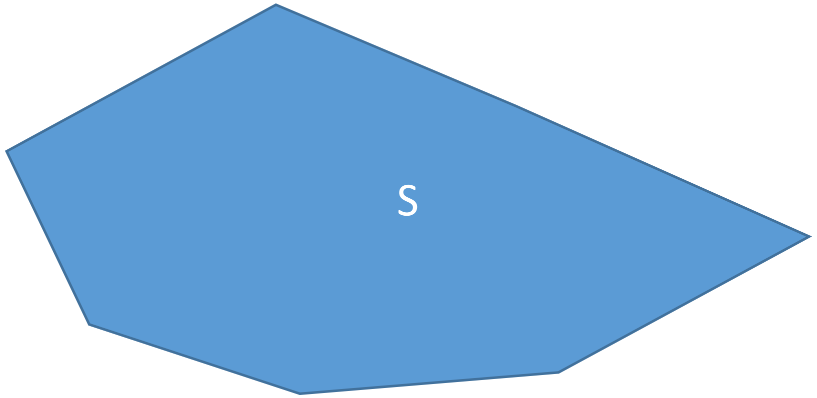
Greenの定理　$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_C (fdx + gdy) = \int_C (f\vec{e}_i + g\vec{e_j}) \cdot \overrightarrow{dr}$

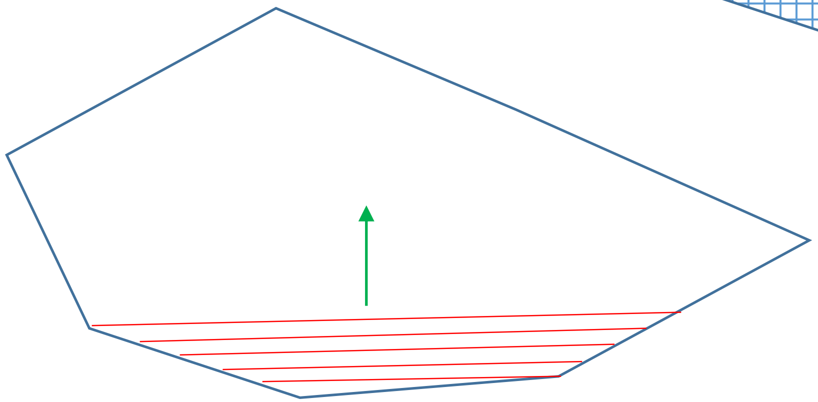
Gaussの発散定理　$\int_S n \times FdS = \int_V \nabla \times FdV$

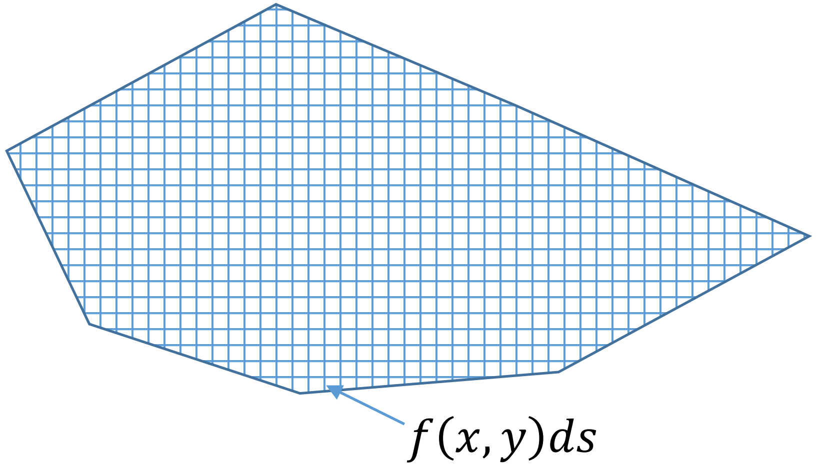
もう一度おさらい

面積積分

$$\int_S f(x,y)dS = \iint f(x,y)dxdy = \int \left[\int f(x,y)dx \right] dy$$





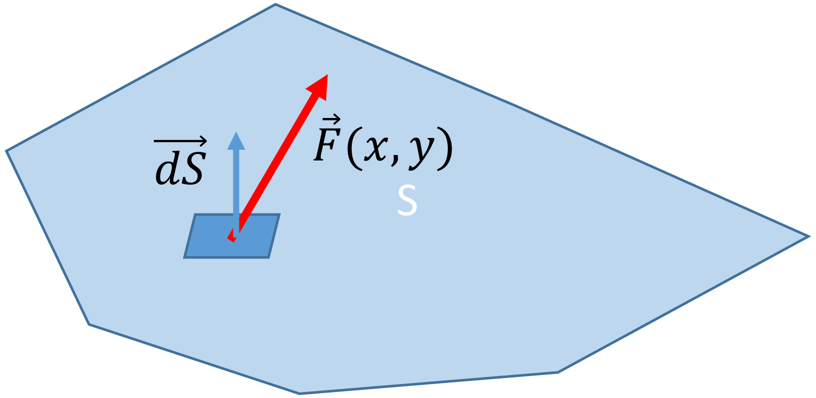


$f(x,y)$　が一定値Aならば

面積×A

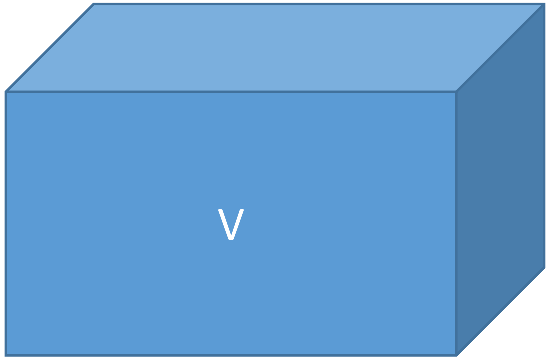
ベクトルだと

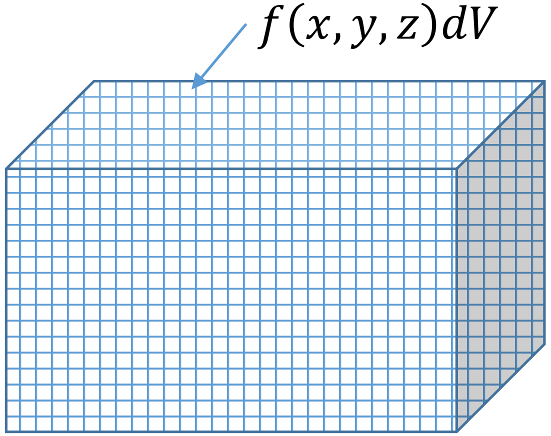
$$\int_S \vec{F}(x,y)\cdot\overrightarrow{dS} = \int_S \vec{F}(x,y)\cdot\vec{n}(x,y)dS \iint \vec{F}(x,y)\cdot\vec{n}(x,y)dxdy = \int \left[\int \vec{F}(x,y)\cdot\vec{n}(x,y)dx \right] dy$$



体積積分

$$\int_V f(x,y,z)dV = \iiint f(x,y,z)dxdydz = \int dz\left[\int dy\left[\int dzf(x,y,z)\right]\right]$$





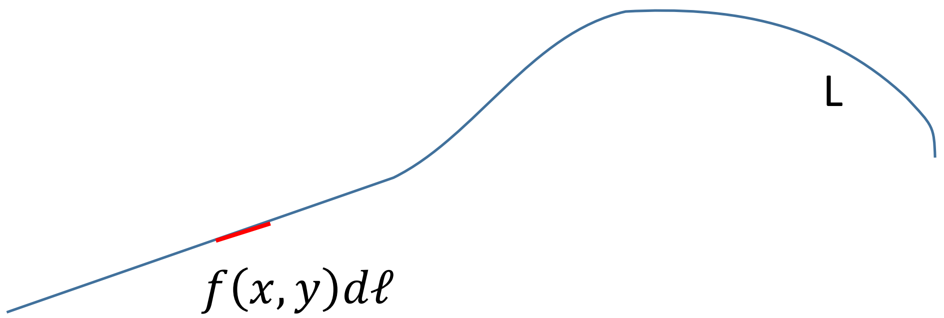
ベクトルだと

$$\int_V \nabla\cdot\vec{A}\,dV = \int_C \vec{A}\cdot\vec{n}\,dS$$

（$\nabla\cdot\vec{A}$ ← スカラー）

線積分

$$\int_L f(x,y)d\ell$$



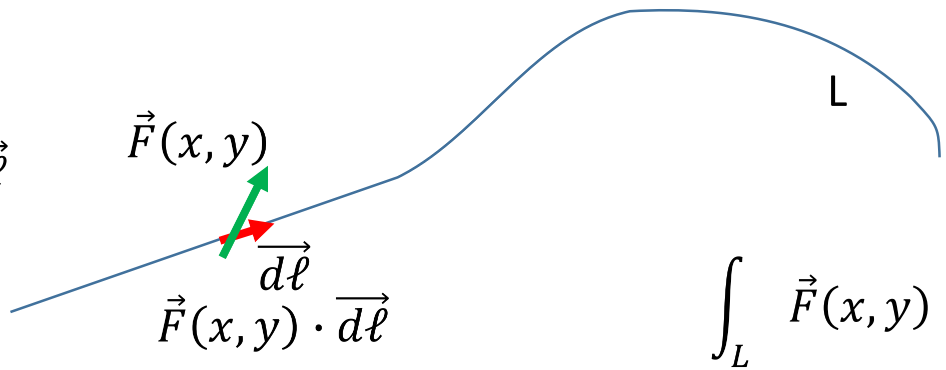
一般にはL上でxが動けばyも変わる

$x(t), y(t)$という媒介変数を使う

$$\int_L f(x,y)d\ell = \int_{t_1}^{t_2} f\big(x(t),y(t)\big)dt$$

ベクトルになると

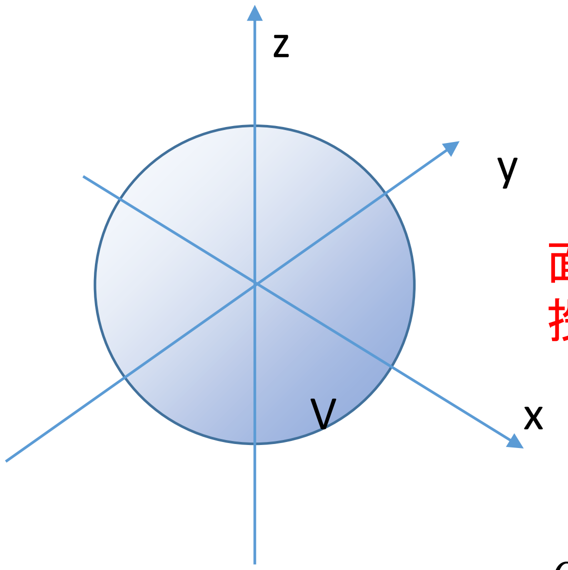
$$\int_L \vec{F}(x,y)\cdot\overrightarrow{d\ell}$$



$$\int_L \vec{F}(x,y)\cdot\overrightarrow{d\ell} = \int_{t_1}^{t_2} \vec{F}(x(t),y(t))\cdot\overrightarrow{s(t)}\,dt$$

接線方向の単位ベクトル

おさらい　Gauss の定理　$\int_V \nabla \cdot \vec{A}\, dV = \int_C \vec{A} \cdot \vec{n}\, dS$



例：　$\mathrm{S}: x^2 + y^2 + z^2 = 1,\ \vec{F} = y\overrightarrow{e_x} - x\overrightarrow{e_y} + z\overrightarrow{e_z}$

Step1: Sの表面の法線ベクトルの関数を求める。

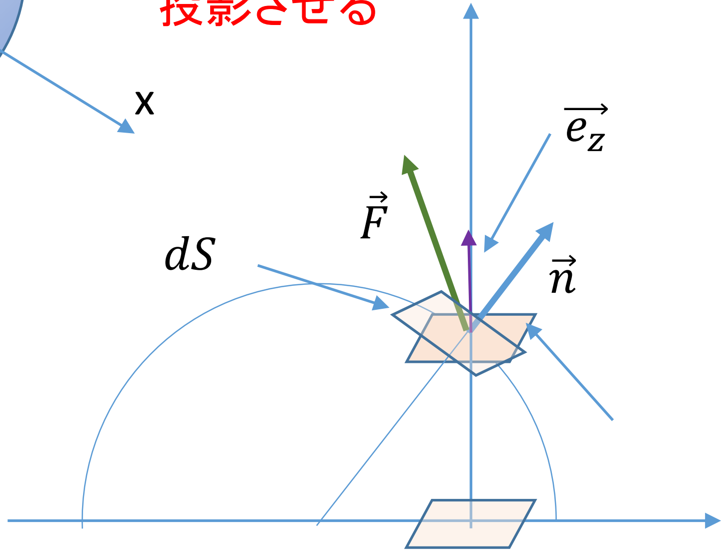
$\mathrm{n} = (xe_x + ye_y + ze_z)$

Step2F · n を計算する。$|\mathrm{n} \cdot \overrightarrow{e_z}|$を求める。

$\mathrm{F} \cdot \mathrm{n} = xy - xy + z^2 = (1 - x^2 - y^2) = (1 - (x^2 + y^2)),$

$|n \cdot e_z| = |z| = \sqrt{1 - (x^2 + y^2)}$

面積分をx, y 平面に投影させる



Step3: $\int_S F \cdot dS = \iint_D F \cdot n \frac{1}{|n \cdot e_z|} dxdy$を計算する。

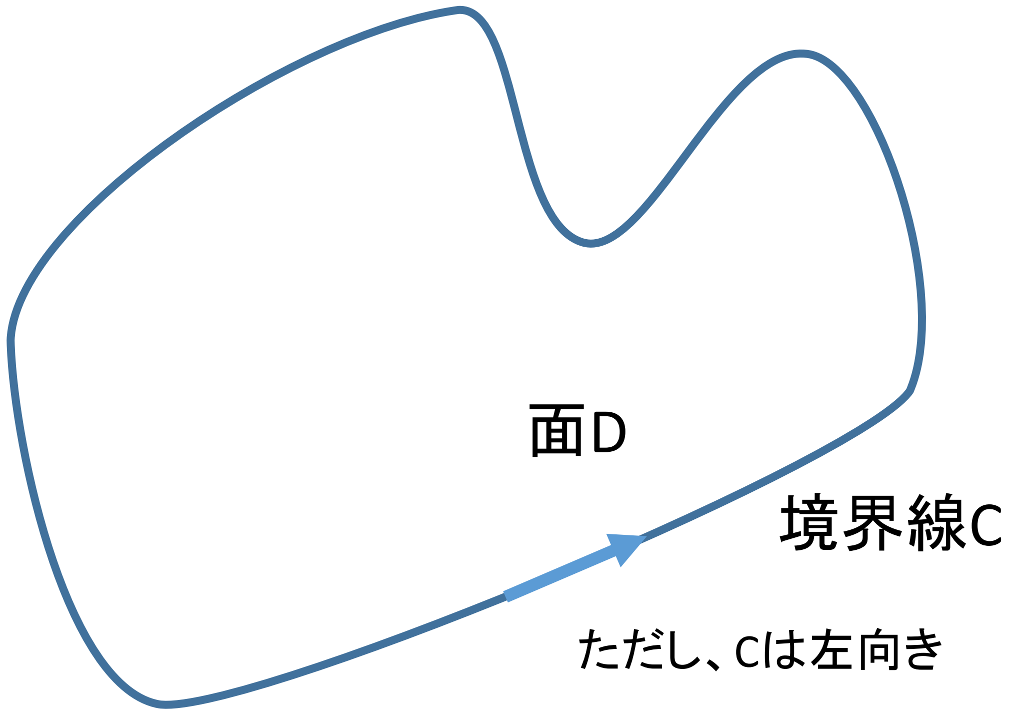
$$\iint_D \frac{1-(x^2+y^2)}{\sqrt{1-(x^2+y^2)}} dxdy = \iint_D \frac{1-r^2}{\sqrt{1-r^2}} d\theta r dr = \int_0^{2\pi} d\theta \int \frac{1-r^2}{\sqrt{1-r^2}} r dr = \pi \int \frac{\xi}{\sqrt{\xi}} d\xi = \pi \left[\frac{2}{3}\xi^{3/2}\right]_0^1 = \frac{2}{3}\pi$$

$$\xi = 1 - r^2, d\xi = -2rdr$$

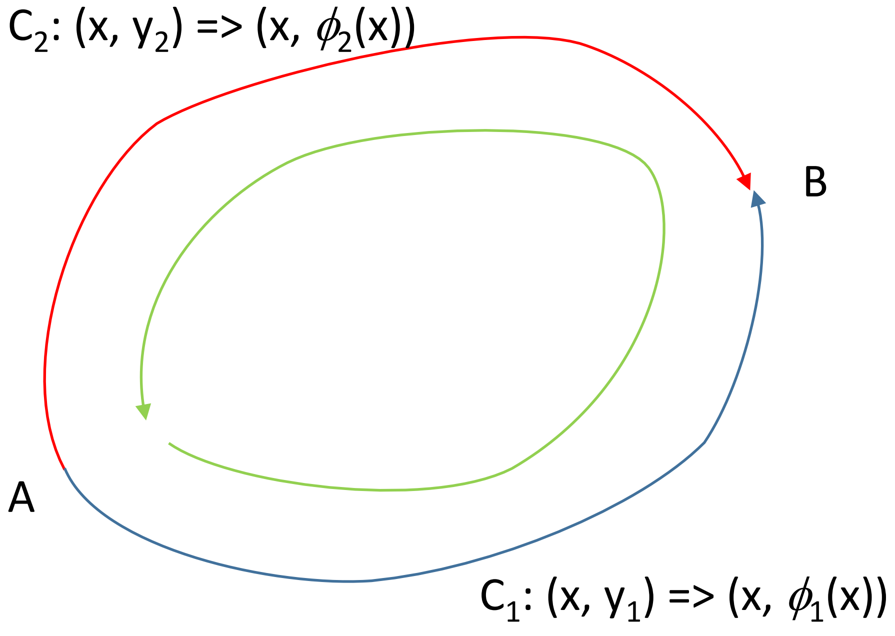
Step4: $\int_V div\vec{F}\, dV$　を計算する。

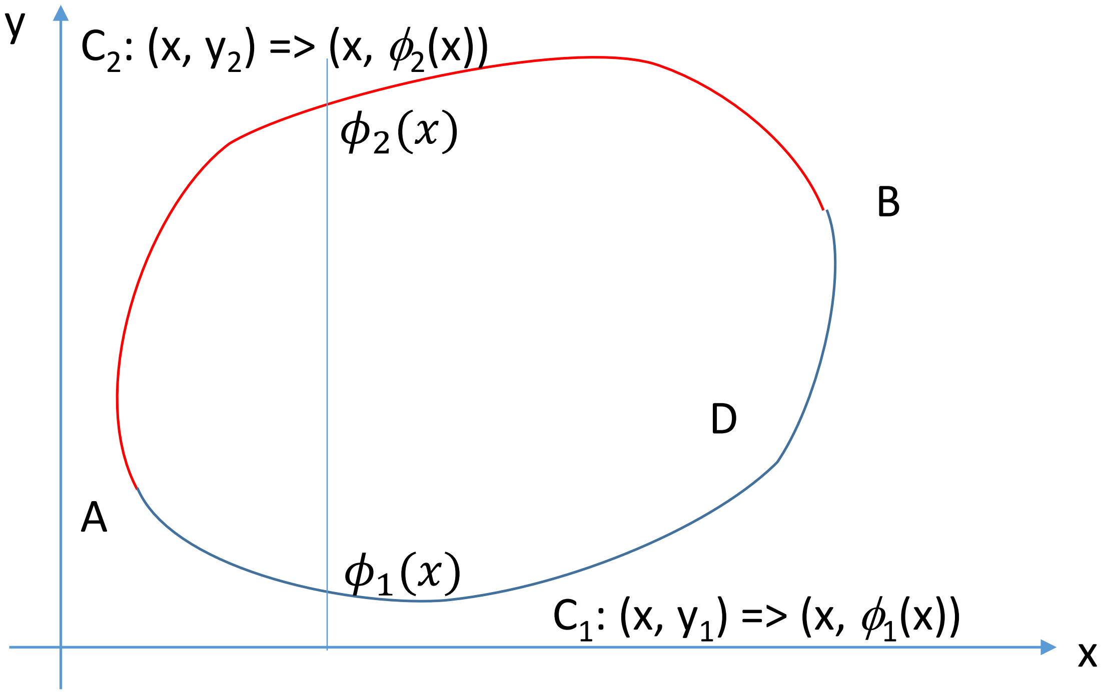
$\mathrm{div\ F} = 0 + 0 + 1 = 1 \ \int_V div\vec{F}\, dV = 1\frac{4\pi}{3} = \frac{4}{3}\pi$

Green の 定理



$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_C (fdx + gdy) = \int_C \left(f\vec{e_i} + g\vec{e_j}\right) \cdot \overrightarrow{dr}$$





$$\int_C f dx = \int_{C1-C2} f dx = \int_{C1} f dx - \int_{C2} f dx$$

$$= \int_A^B f(x, y_1(x)) dx - \int_A^B f(x, y_2)(x) dx$$

$$= \int_A^B f(x, \phi_1(x)) dx - \int_A^B f(x, \phi_2(x)) dx$$

$$\iint_D \left(\frac{\partial f}{\partial y}\right) dxdy = \int_A^B \left(\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial y} dy\right) dx = \int_A^B [f]_{\phi_1(x)}^{\phi_2(x)} dx$$

$$= \int_A^B (f(x, \phi_2(x)) - f(x, \phi_1(x))) dx$$

$$= \int_A^B f\big(x, \phi_2(x)\big) dx - \int_A^B f\big(x, \phi_1(x)\big) dx$$

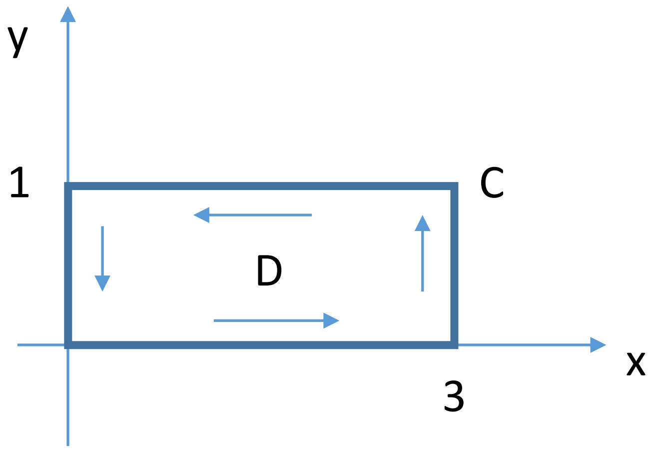
$$\int_C f dx = -\iint_D \left(\frac{\partial f}{\partial y}\right) dxdy$$

$$\int_C g dy = -\iint_D \left(\frac{\partial g}{\partial x}\right) dxdy$$

Greenの定理

$$\int_C f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$\int_C f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$



$$f(x, y) = x^3 - 3x^2 y, \qquad g(x, y) = 2xy + y^2$$

$$\frac{\partial f(x, y)}{\partial y} = -3x^2, \frac{\partial g(x, y)}{\partial x} = 2y$$

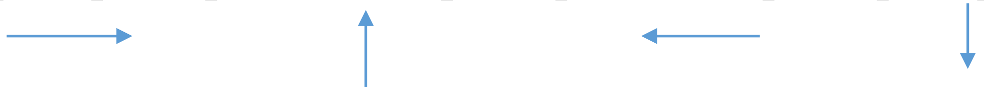
$$\int_C (f dx + g dy) = \int_C (x^3 - 3x^2 y) dx + (2xy + y^2) dy$$ を上の周で計算しようとする

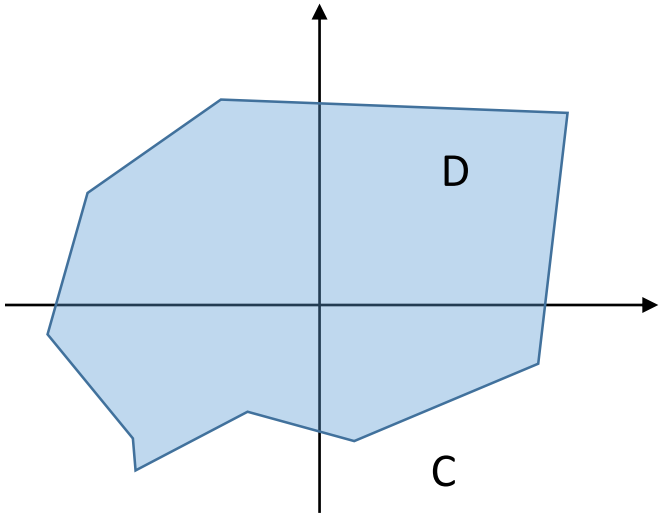
グリーンの公式を使えば、

$$\iint_D (2y + 3x^2) dxdy = \int_0^3 [y^2 + 3x^2 y]_0^1 dx = \int_0^3 (1 + 3x^2) dx = [x + x^3]_0^3 = 30$$

$$\int_C (x^3 - 3x^2 y) dx + (2xy + y^2) dy$$

$$= \left[\frac{1}{4}x^4\right]_0^3 + \left[3y^2 + \frac{1}{3}y^3\right]_0^1 + \left[\frac{1}{4}x^4 - x^3\right]_3^0 + \left[\frac{1}{3}y^3\right]_1^0 = \frac{81}{4} + 3 + \frac{1}{3} - \frac{81}{4} + 27 - \frac{1}{3} = 30$$

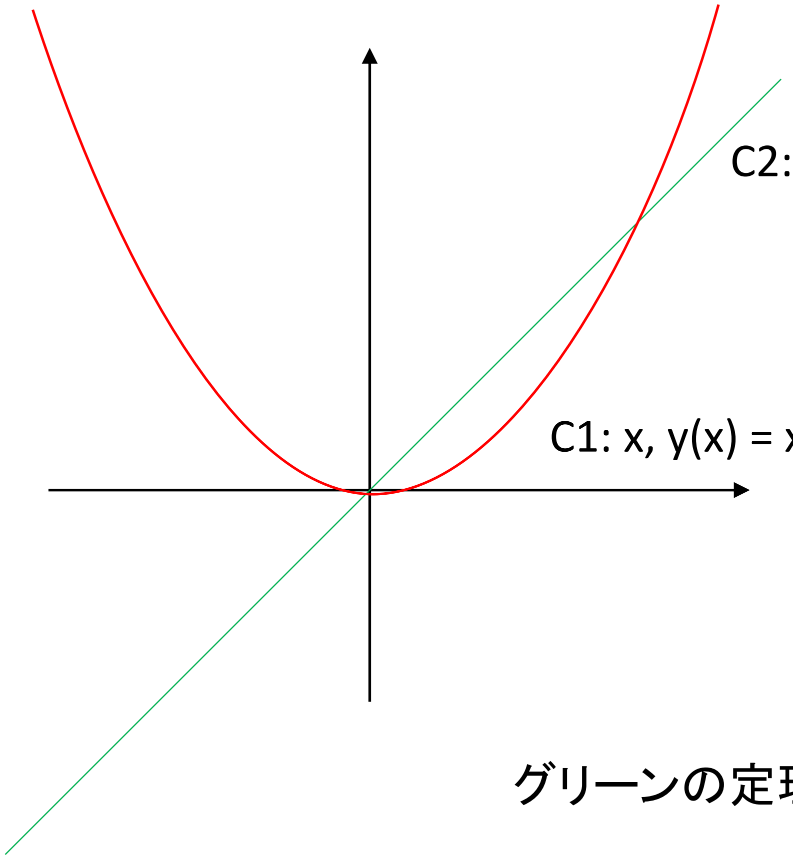




$$\iint_D dxdy$$ 面積

$$\int_C ydx - xdy = \iint_D \left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right) dxdy = -2 \iint_D dxdy$$

線積分で面積？



$$I=\int_C (xy+x^2)dx+x^2dy$$

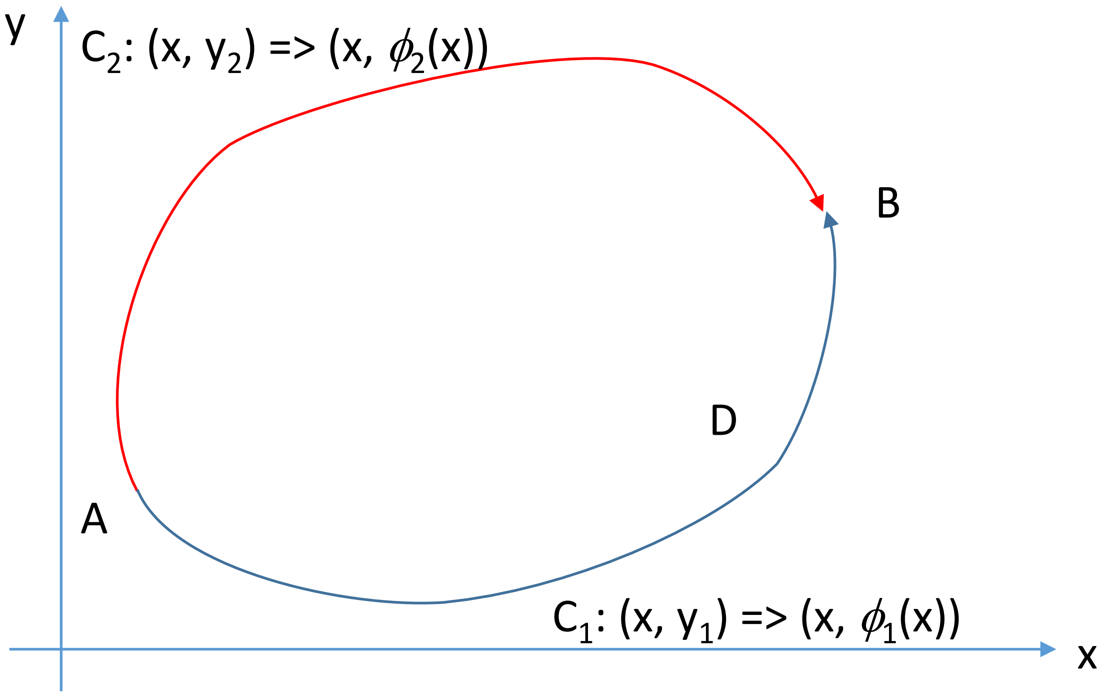
$$\int_{C1} (xy+x^2)dx+x^2dy-\int_{C2} (xy+x^2)dx+x^2dy$$

$$\int_0^1 (x^3+x^2+2x^3)dx-\int_0^1 (x^2+x^2+x^2)dx=\frac{1}{12}$$

$$dy=2xdx \qquad dy=dx$$

グリーンの定理を使うと

$$\iint_D \left(\frac{\partial x^2}{\partial x}-\frac{\partial}{\partial y}(xy+x^2)\right)dxdy=\iint_D xdxdy=\int_0^1\left(\int_{x^2}^{x} xdy\right)dx=\int_0^1 [xy]_{x^2}^{x}dx=\int_0^1 x(x-x^2)dx=\frac{1}{12}$$



$$\int_C f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$\int_C f dx + g dy = \int_{C1} f dx + g dy - \int_{C2} f dx + g dy$$

もし $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$ ならば

$f dx + g dy$ は経路によらずに同じ値をとる。

＝＞ポテンシャルみたいなもの＝＞Ψ(x,y)

$$\int_{C1,C2} f dx + g dy = \Psi(B) - \Psi(A)$$

ここで、 $d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy$

$$\Psi(B) - \Psi(A) = \int_A^B d\Psi = \int_A^B \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy$$

$\frac{\partial \Psi}{\partial x} \rightarrow f$, $\frac{\partial \Psi}{\partial y} \rightarrow g$

例えば、 $\Psi = x^3 y + 2y^2$ とすれば、 $f = \dfrac{\partial \Psi}{\partial x} = 3x^2 y,$ $g = \dfrac{\partial \Psi}{\partial y} = x^3 + 4y$ の時

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3x^2 - 3x^2 = 0$$

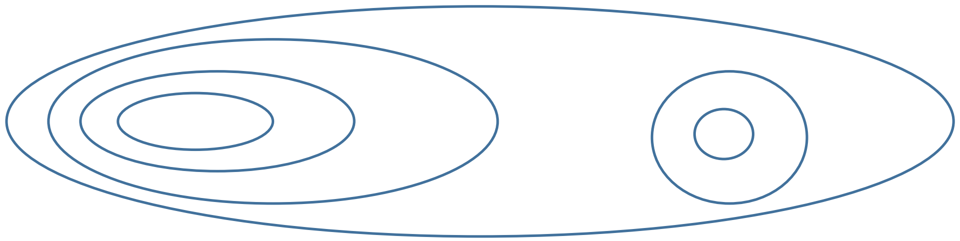
$$\int_C f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

から、経路によらない

$$\frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0$$

$\Psi(x, y)$ =const.

等高線

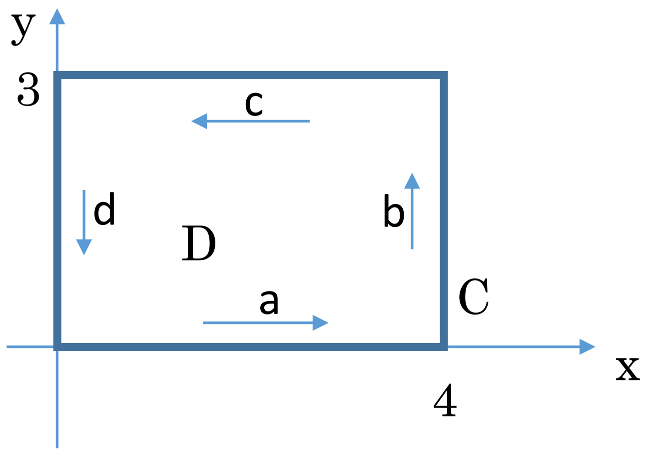


確認:グリーンの定理

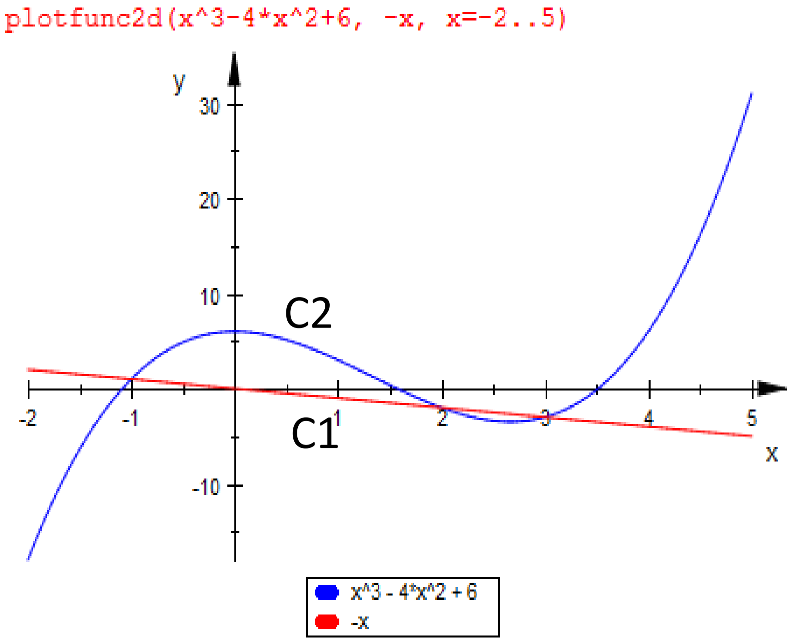
$$\iint_D \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy=\int_C (fdx+gdy)=\int_C (f\vec{e_i}+g\vec{e_j})\cdot\overrightarrow{dr}$$

問題１　$\mathrm{f(x,y)}=x^2+y^2,\quad g(x,y)=2xy$

で、Greenの定理を検証



問題２　C1:y = -x, C2:y=x³-4x²+6 の２つの線はx=-1, 2, 3で交点をとる。このx=-1 ～ x=2 の範囲で囲まれる閉曲線Cで、$\mathrm{f}=\mathrm{xy},\mathrm{g}=x^2$としたときに、$\mathrm{I}=\int_C fdx+gdy$ の値を計算しなさい。



問題1

$$\frac{\partial g}{\partial x} = 2\mathrm{y}$$

$$\frac{\partial \mathrm{f}}{\partial \mathrm{y}} = 2y$$

a の経路 y=0、x = 0 ~ 4 $\int_C (fdx + gdy) = \left[\frac{1}{3}x^3 + y^2 x\right]_0^4 = \frac{64}{3}$

b の経路 x=4 $= [xy^2]_0^3 = 4 \times 9 = 36$

c の経路 y=3 $= \left[\frac{1}{3}x^3 + y^2 x\right]_4^0 = -\frac{64}{3} - 36$

d の経路でx=0 $= [xy^2]_3^0 = 0$

$$= \frac{64}{3} + 36 - \frac{64}{3} - 36 + 0 = 0$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2\mathrm{y} - 2\mathrm{y} = 0$$

問題2

$$\int_C f dx + g dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \iint_D (2x - x) dxdy$$

$$= \iint_D x dxdy = \int_{-1}^{2} \left(\int_{c1}^{c2} x dy\right) dx$$

$$= \int_{-1}^{2} \left([xy]_{-x}^{x^3 - 4x^2 + 6}\right) dx = \int_{-1}^{2} (x^4 - 4x^3 + 6x - (-x^2)) dx$$

$$= \left[\frac{1}{5}x^5 - x^4 + \frac{1}{3}x^3 + 3x^2\right]_{-1}^{2} = \frac{76 - 22}{15} = \frac{18}{5}$$