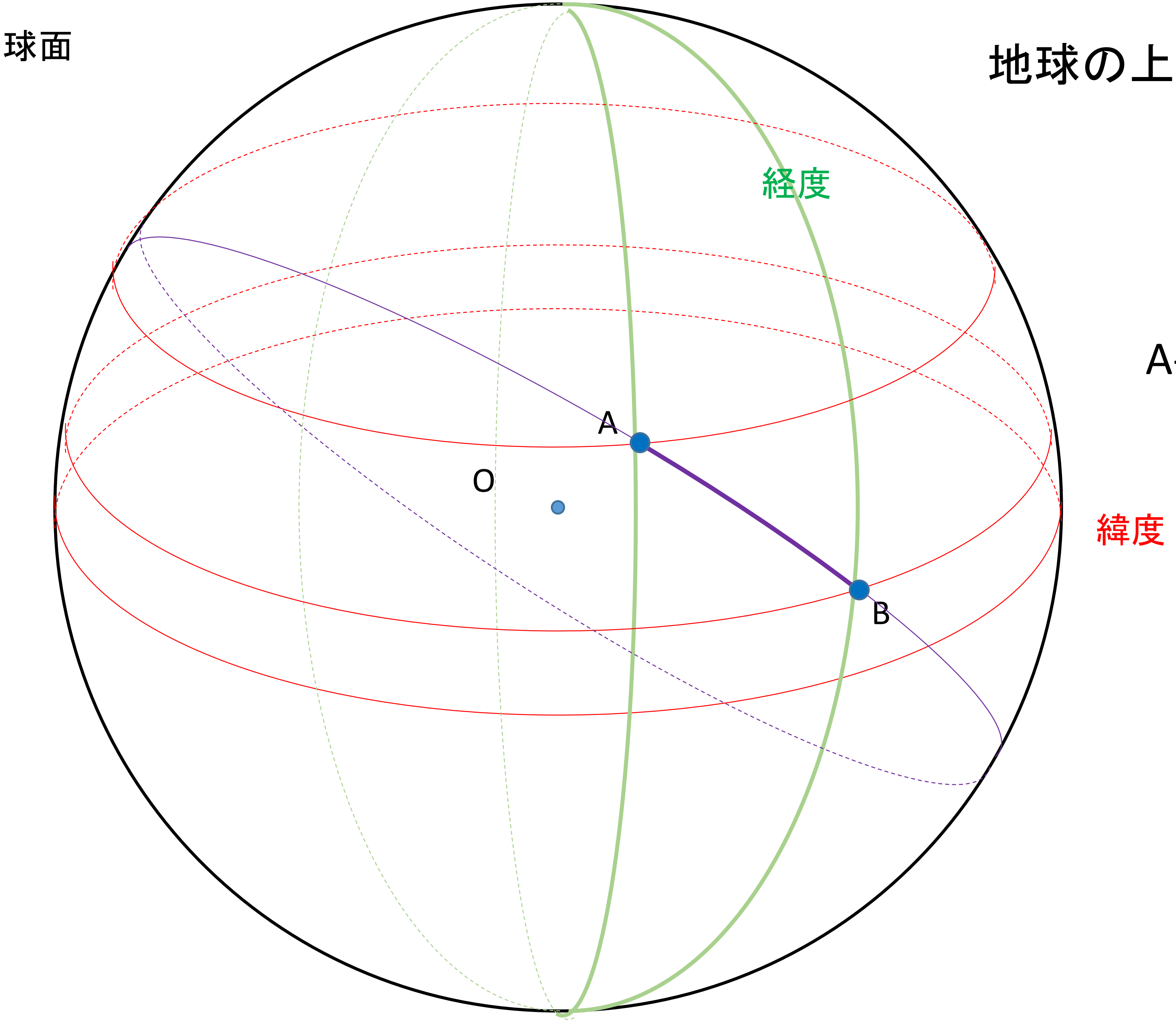
米田、戸倉川、月7限1930~2100、西5-109

応用数学 A 第8回

球面三角法
三角関数の複素数化

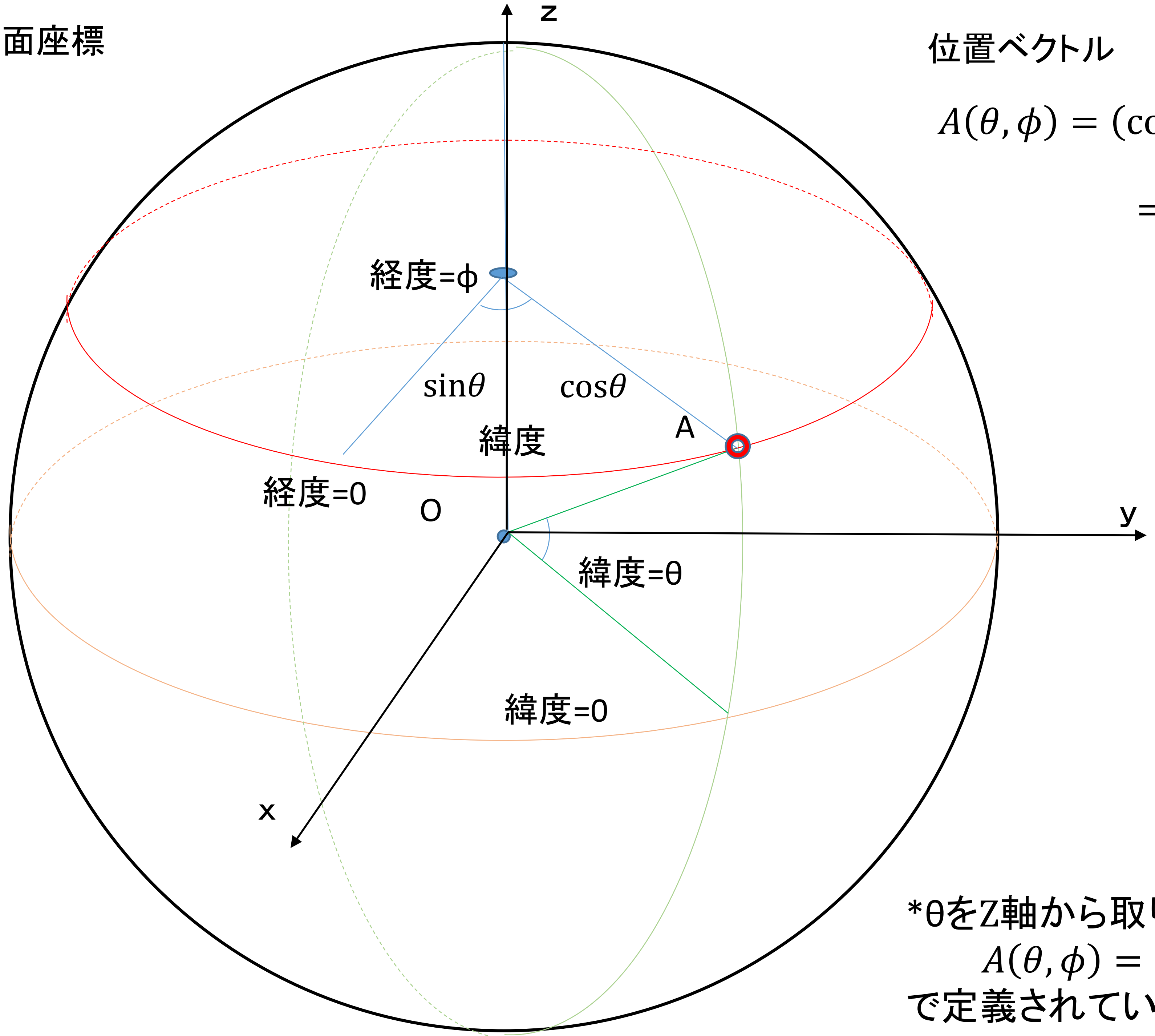
http://gt_ils.ils.uec.ac.jp/AppMath/

地球の上での球面上の座標



A-B間の距離は？

球面座標



位置ベクトル

$$A(\theta,\phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

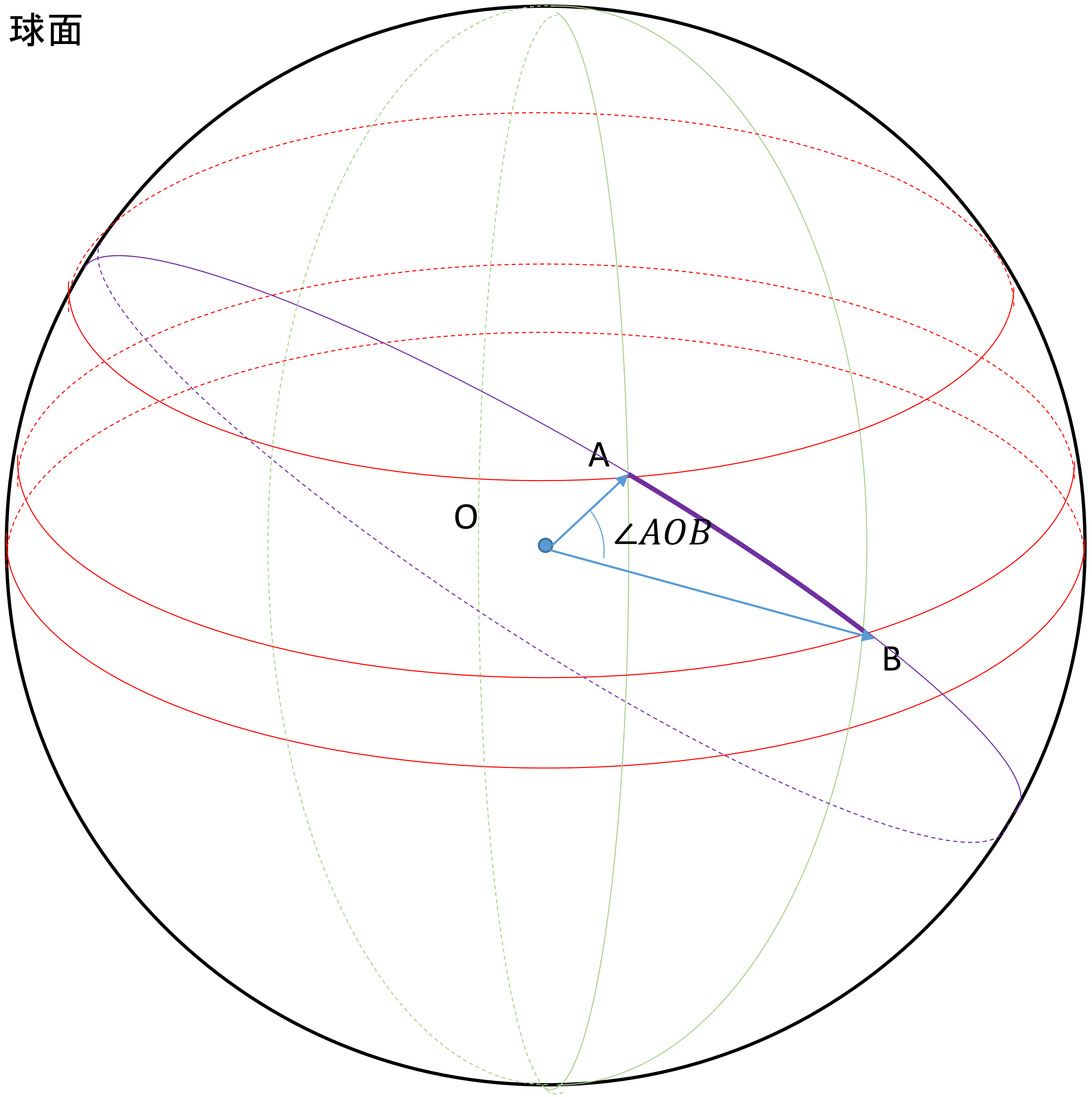
$$= \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$$

*θをZ軸から取り

$$A(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

で定義されている教科書も多い

球面



地球の上での球面上の座標

$$\cos\angle AOB = \overrightarrow{OA}\cdot\overrightarrow{OB}$$

$$\overrightarrow{OA} = (\cos\theta_A\cos\phi_A, \cos\theta_A\sin\phi_A, \sin\theta_A)$$

$$\overrightarrow{OB} = (\cos\theta_B\cos\phi_B, \cos\theta_B\sin\phi_B, \sin\theta_B)$$

内積とって角度を求めて、

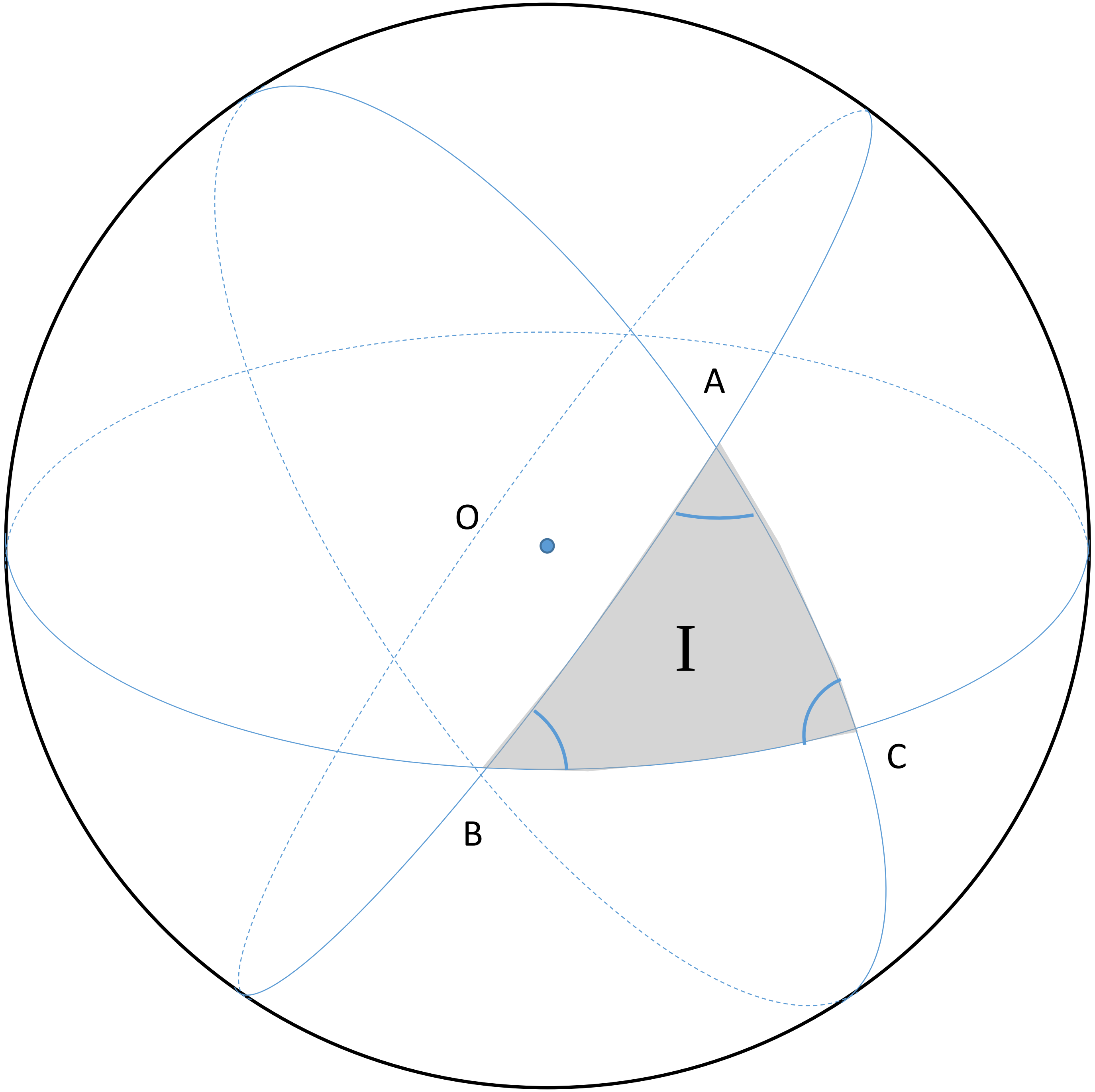
地球の円周(4万km)× $\frac{\angle AOB}{2\pi}$

東京：E139.75° N35.68°
ミラノ：E9.19° N45.46°

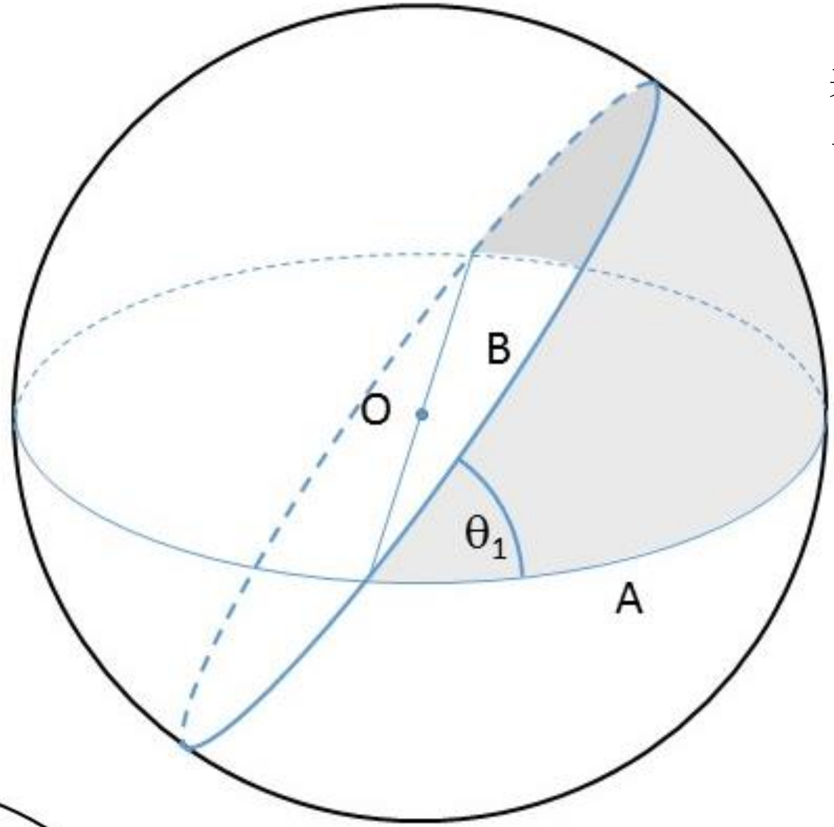
$$\cos\angle AOB = 154.77$$

東京-ミラノ：17197.27km

球面三角法

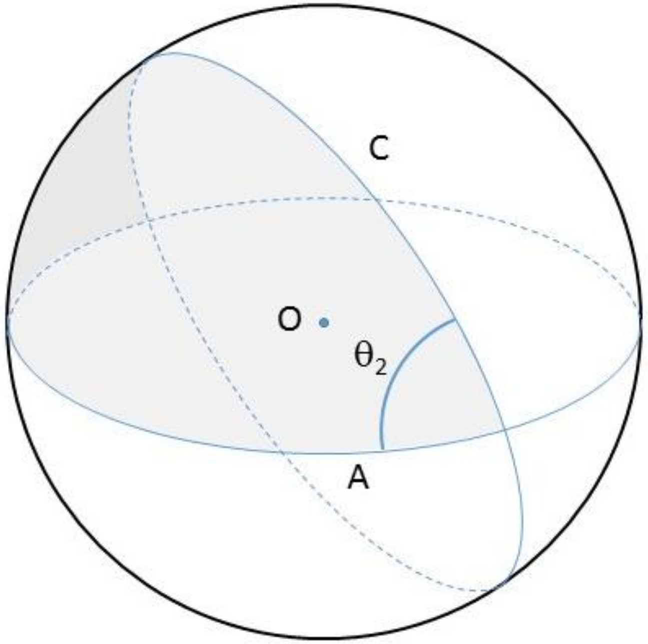


Step 1



表面積S_{AB}は、θ_1に比例することは明らかで、$\theta_1 = \pi$ で半球の表面積($2\pi r^2$)に等しくなる。

$$S_{AB} = 2\pi r^2 \frac{\theta_1}{\pi}$$



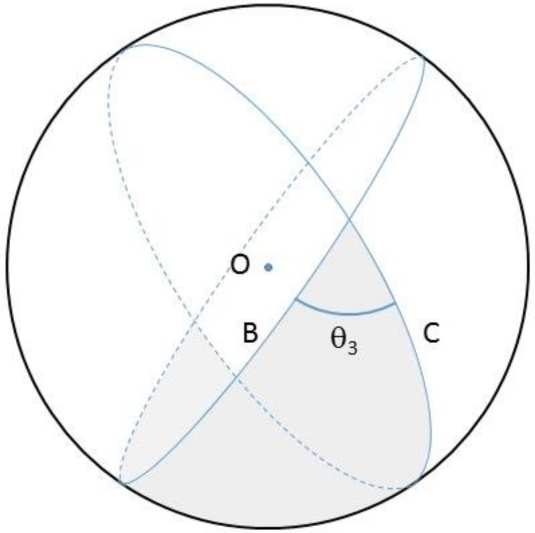
Step 2

同様に、赤道線A, Cで囲まれた面積S_{AC}も、その角度θ_2とすれば

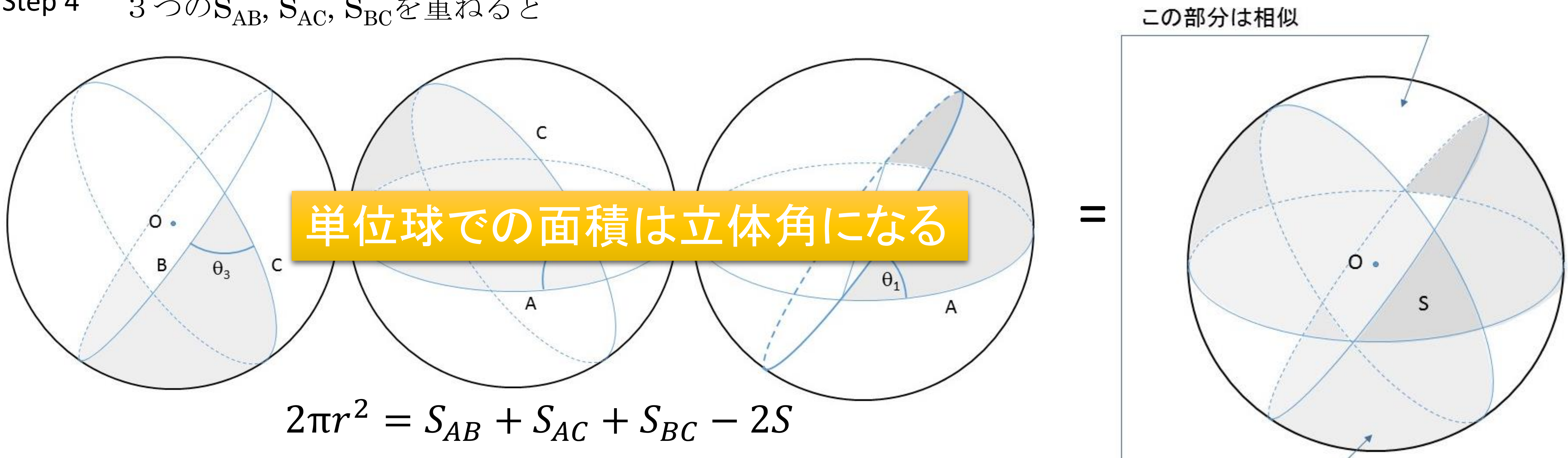
$$S_{AC} = 2\pi r^2 \frac{\theta_2}{\pi}$$

Step 3　赤道線B, Cで囲まれた面積S_{BC}も、その角度θ_3 とすれば

$$S_{BC} = 2\pi r^2 \frac{\theta_3}{\pi}$$



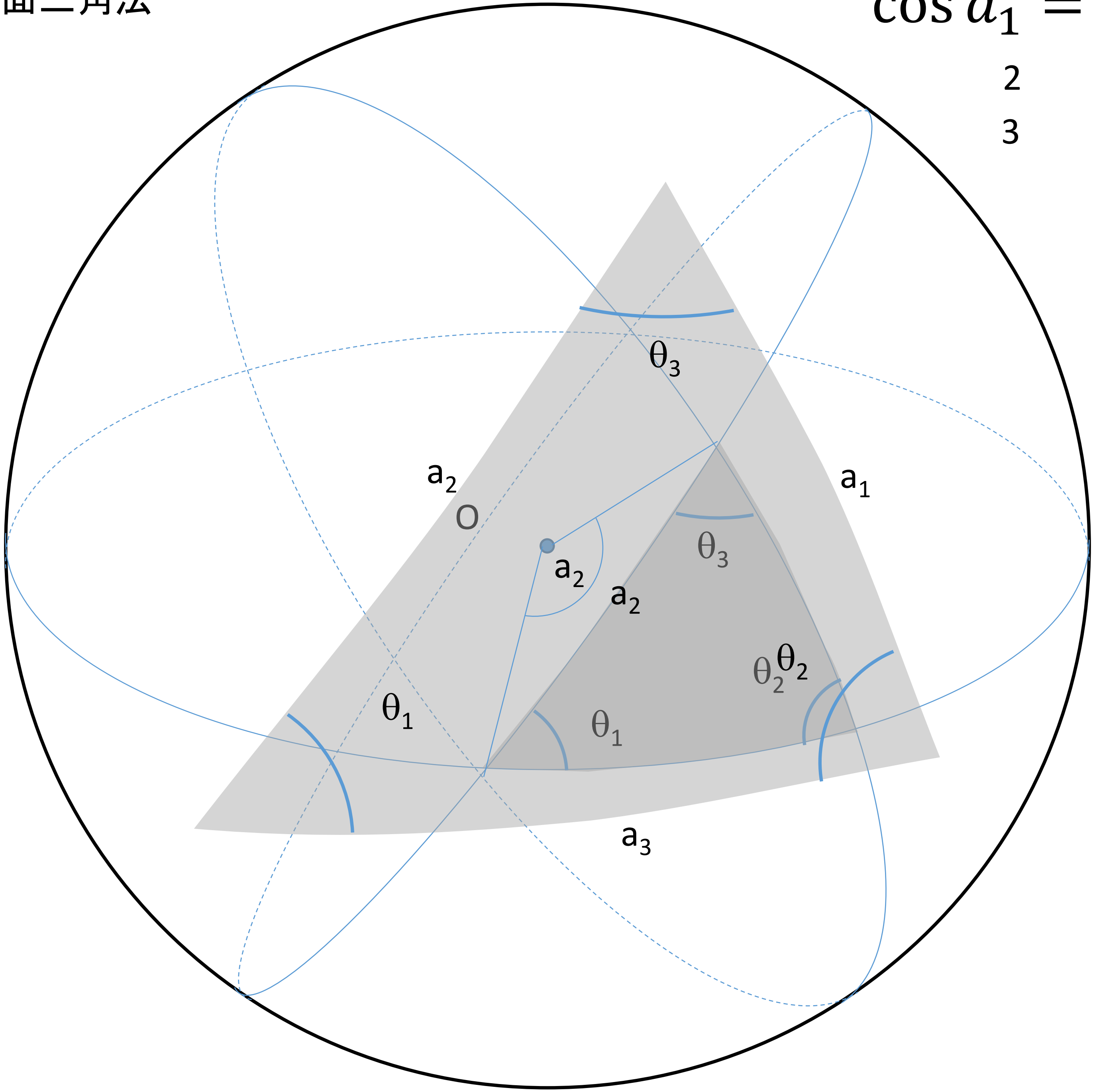
Step 4　３つのS_{AB}, S_{AC}, S_{BC}を重ねると



$$2\pi r^2 = S_{AB} + S_{AC} + S_{BC} - 2S$$

$$S = \pi r^2 \frac{\theta_1}{\pi} + \pi r^2 \frac{\theta_2}{\pi} + \pi r^2 \frac{\theta_3}{\pi} - \pi r^2 = (\theta_1 + \theta_2 + \theta_3 - \pi) r^2$$

球面三角法



余弦定理

$$\cos a_1 = \cos a_2 \cos a_3 + \sin a_2 \sin a_3 \cos \theta_1$$

2 3 1 3 1 2

3 1 2 1 2 3

正弦定理

$$\frac{\sin \theta_1}{\sin a_1} = \frac{\sin \theta_2}{\sin a_2} = \frac{\sin \theta_3}{\sin a_3}$$

球面余弦定理の証明

$$\overline{DE}^2 = \overline{AD}^2 + \overline{AE}^2 - 2\overline{AD}\cdot\overline{AE}\cos A$$
$$= \overline{OD}^2 + \overline{OE}^2 - 2\overline{OD}\cdot\overline{OE}\cos a$$

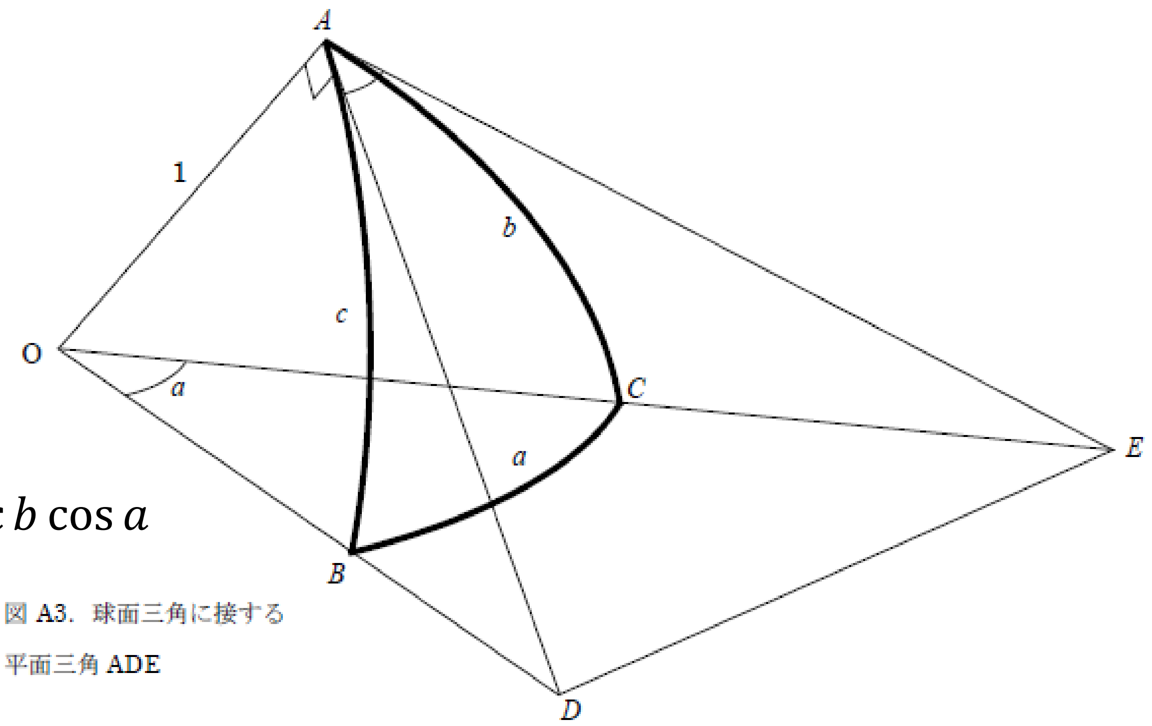


図 A3. 球面三角に接する
平面三角 ADE

http://www.astro.sci.yamaguchi-u.ac.jp/~kenta/eclipse/SphericalTriangle081106.pdf

$$\overline{AD} = \tan c$$
$$\overline{AE} = \tan b$$
$$\overline{OD} = 1/\cos c$$
$$\overline{OE} = 1/\cos b$$

$$\tan^2 c + \tan^2 b - 2\tan c\tan b\cos A = \sec^2 c + \sec^2 b - 2\sec c\sec b\cos a$$

$$\begin{cases} \tan^2 c + \tan^2 b = \dfrac{\sin^2 c}{\cos^2 c} + \dfrac{\sin^2 b}{\cos^2 b} \\ \sec^2 c + \sec^2 b = \dfrac{1}{\cos^2 c} + \dfrac{1}{\cos^2 b} = \dfrac{\sin^2 c + \cos^2 c}{\cos^2 c} + \dfrac{\sin^2 b + \cos^2 b}{\cos^2 b} \end{cases}$$

$$\therefore \left(\sec^2 c + \sec^2 b\right) - \left(\tan^2 c + \tan^2 b\right) = 2$$

$$\cos a = \frac{1}{\sec c\sec b} + \frac{\tan c\tan b\cos A}{\sec c\sec b}$$
$$= \cos b\cos c + \sin b\sin c\cos A$$

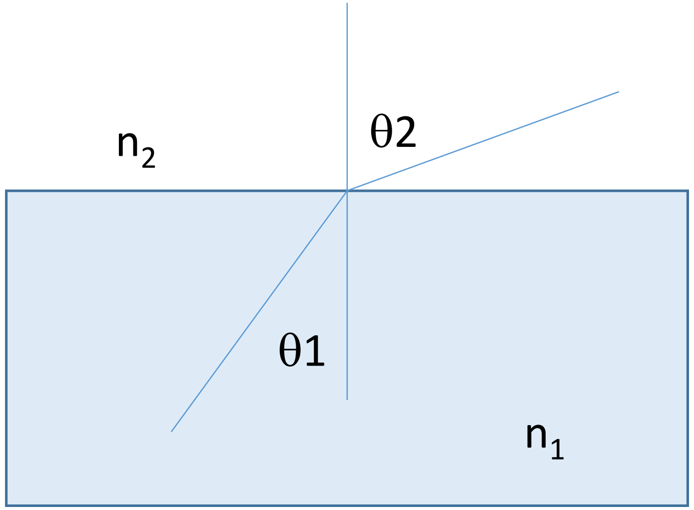
$$\cos a_1 = \cos a_2\cos a_3 + \sin a_2\sin a_3\cos\theta_1$$

複素数：complex number

X=α+βi

なぜ、複素数？

２次方程式？



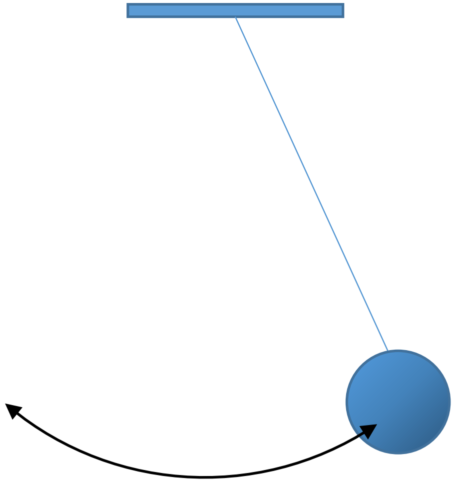
スネルの法則

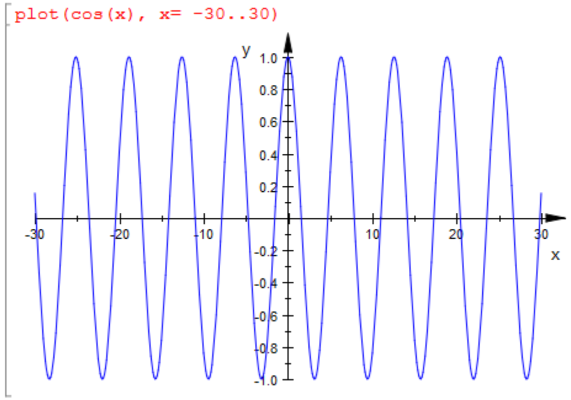
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

もし、n_2=1 で$n_1 \sin\theta_1$ ＞ 1だったら？
$\sin\theta_2$ =1.5の答えは？

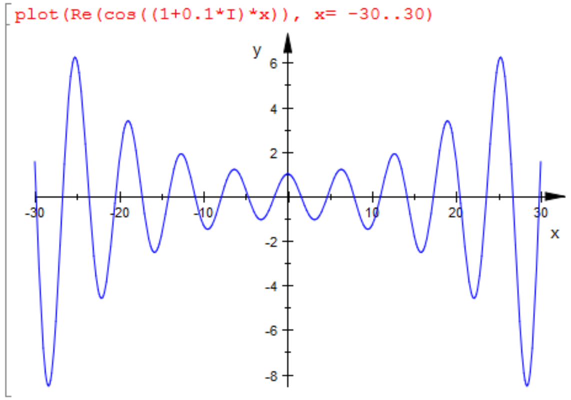
$$\theta_2 = 2\pi m + \frac{\pi}{2} + 0.9624\, i$$

全反射の時の
Evanescent wave





$\cos x$



$\cos(1 + 0.1i)x$

オイラーの公式

テーラー展開
(マクローリン展開)

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots..$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} +$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} +$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} -$$

x^2 おきに-1

近似式ここから
X<<1ならば
$(1+x)^n = 1+nx$
Sin(x)=x

$$e^{ix} = 1 + ix + i^2\frac{x^2}{2!} + i^3\frac{x^3}{3!} + i^4\frac{x^4}{4!} + i^5\frac{x^5}{5!} + i^6\frac{x^6}{6!} + i^7\frac{x^7}{7!} + i^8\frac{x^8}{8!} +$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots. + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots\right)$$

$$= \cos x + i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

逆三角関数

$$\sin x = a = \frac{e^{ix} - e^{-ix}}{2i}$$ となるxの値を求める関数 $$e^{2ix} - 2iae^{ix} - 1 = 0$$

$$e^{ix} = ia + \sqrt{-a^2 + 1}$$ $$x = \frac{1}{i}\log(ia + \sqrt{1 - a^2})$$ $$\arcsin(a), \sin^{-1}(a)$$

$$\cos x = a = \frac{e^{ix} + e^{-ix}}{2}$$ となるxの値を求める関数 $$e^{2ix} - 2ae^{ix} + 1 = 0$$

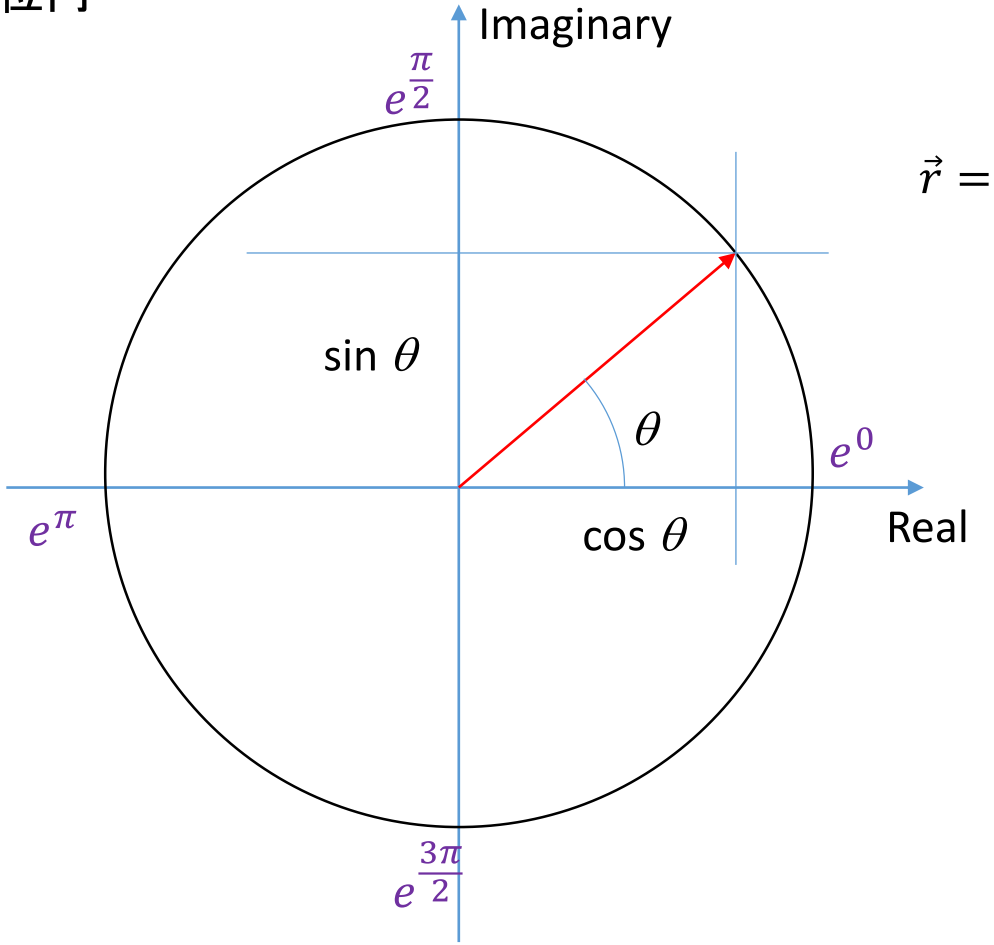
$$e^{ix} = a + \sqrt{a^2 - 1}$$ $$x = \frac{1}{i}\log(a + \sqrt{a^2 - 1})$$ $$\arccos(a), \cos^{-1}(a)$$

a>1でxは純虚数

$$\tan x = a = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$ となるxの値を求める関数 $$\frac{e^{2ix} - 1}{e^{2ix} + 1} = ia$$

$$e^{2ix} = \frac{1 + ia}{1 - ia} = \frac{i - a}{i + a}$$ $$x = \frac{1}{2i}\log(\frac{i - a}{i + a})$$ $$\arctan(a), \tan^{-1}(a)$$

複素面、単位円



$\vec{r} = \cos\theta + \mathrm{i}\cdot\sin\theta$

Y=Log(-1)

Y=?

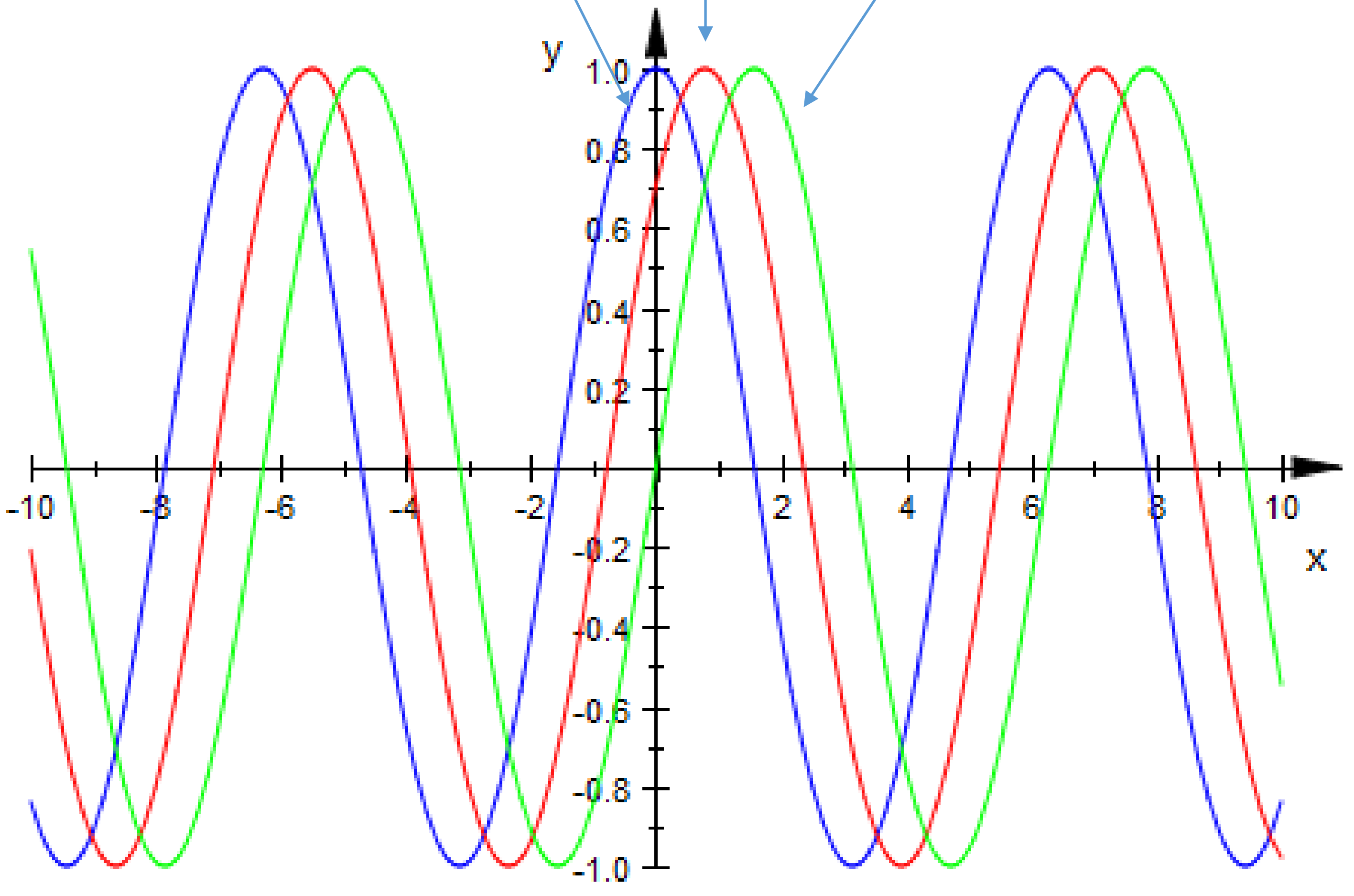
e^{Y}=-1

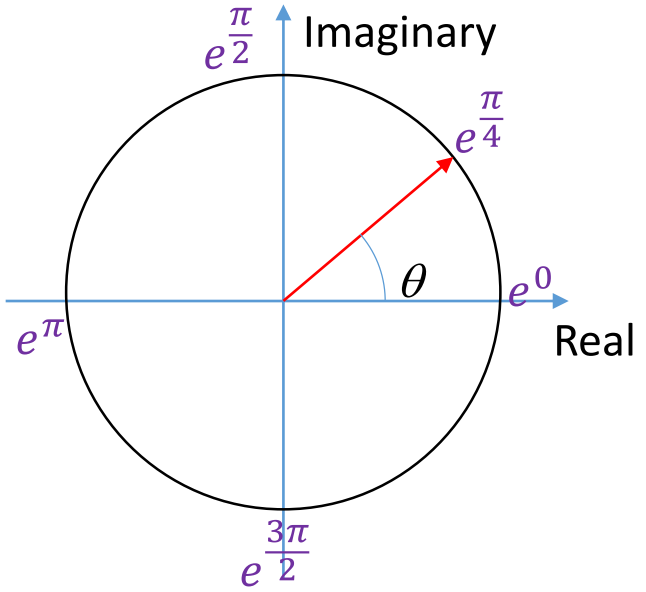
Y=iπ(1+2n)

位相という考え

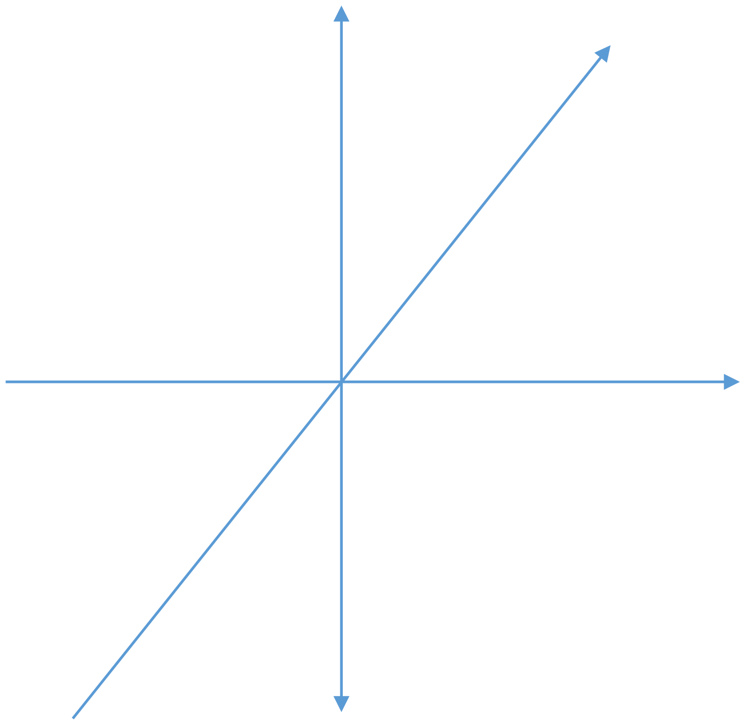
e^{ix} $e^{i(x-\frac{\pi}{4})}$ $e^{i(x-\frac{\pi}{2})}$

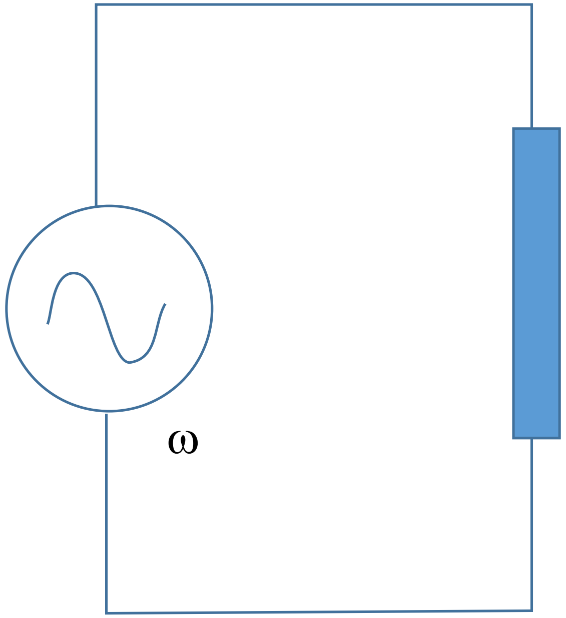
```
plot(Re(exp(I*x)), Re(exp(I*(x-PI/4))), Re(exp(I*(x-PI/2))), x= -10..10)
```

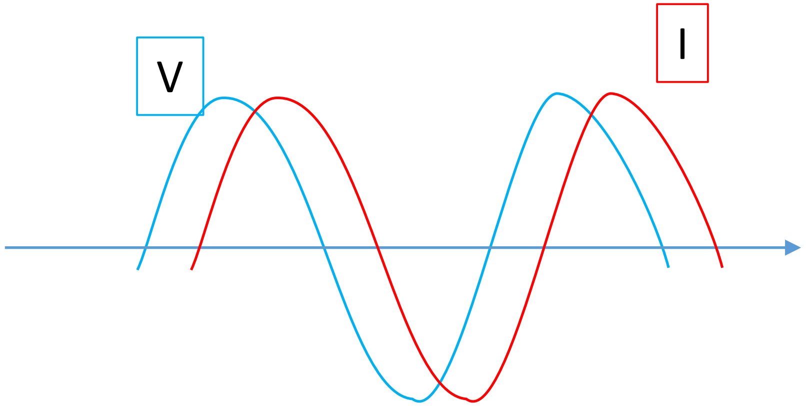




力率という考え







$$\int_{-\infty}^{\infty}\sin(wt)\sin(wt+\pi/2)\mathrm{dt}=\int\ \sin(wt)cos(wt)\mathrm{dt}$$

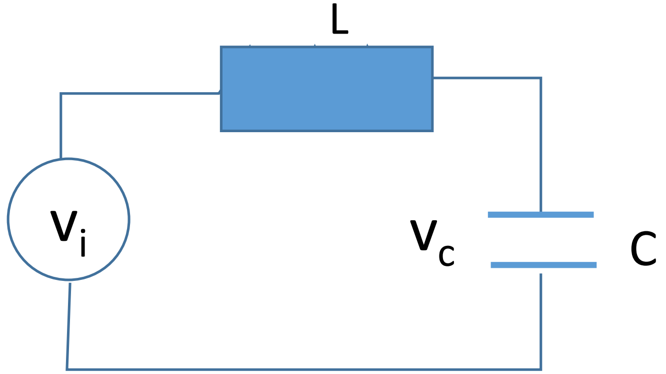
$$=\frac{1}{2}\int\ \sin(2wt)\mathrm{dt}$$

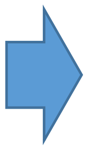
$$=0$$

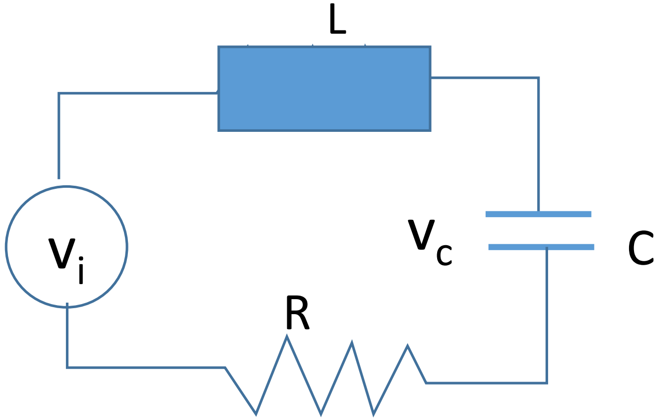
微分方程式の解

$$L\frac{dI}{dt}+\frac{1}{C}\int Idt=V$$

$$L\frac{dI}{dt}+\frac{1}{C}\int Idt+RI=V$$





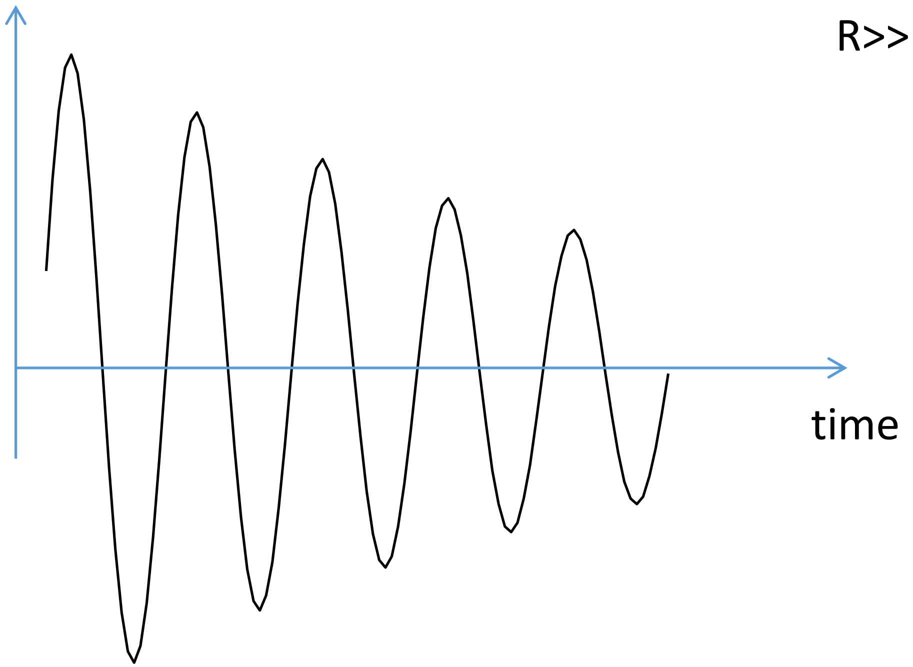


LC 回路では$1/\sqrt{LC}$の周波数を持つ振動解

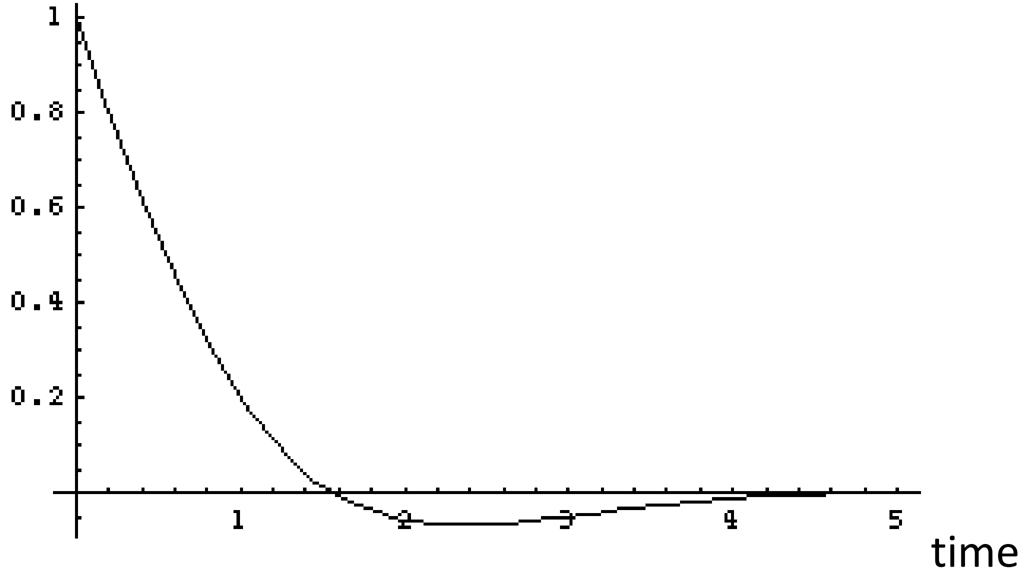
$$I(t)=A_0\exp(iw_0t)+B_0\exp(iw_0t)$$

LCRでは条件により振動減衰

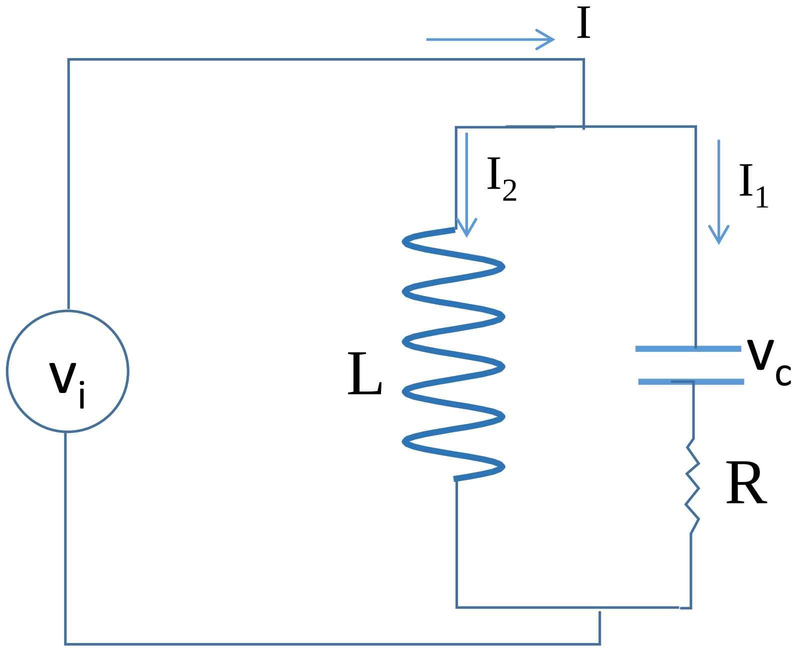
R<<L/C



R>>L/C



損失項



$$\frac{1}{C}\int I_1 dt + RI_1 = L\frac{\partial I_2}{\partial t}$$

$$I = I_1 + I_2$$

$$I = I_0 e^{-i\omega t}$$

$$L\ddot{I_1} + R\dot{I_1} + \frac{1}{C}I_1 = -\omega^2 I_0 e^{-i\omega t}$$

もし、Rがなければ、

$$L\ddot{I_1} + \frac{1}{C}I_1 = -\omega^2 I_0 e^{-i\omega t}$$

$$I = I_0 \sin\omega t$$

入力電源を切ると？

$$L\ddot{I_1} + R\dot{I_1} + \frac{1}{C}I_1 = 0$$

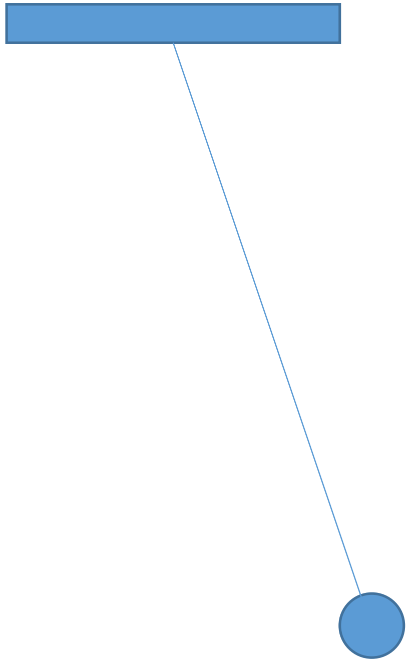
$I_1 = Ae^{-i\omega_1 t}$として

$$-L\omega_1{}^2 - \omega_1 R + \frac{1}{C} = 0$$

$$\omega_1 = \frac{-i\frac{R}{L} \pm \sqrt{-\left(\frac{R}{L}\right)^2 + 4\frac{1}{LC}}}{2}$$

$$R^2 \ll \frac{4}{LC} \qquad \omega_1 = \sqrt{\frac{1}{LC} - \left(\frac{R}{4L}\right)^2} - i\frac{R}{2L}$$

Wの虚数成分の意味　exp(-i*iR/2Lt)=exp(-R/2Lt)損失項となる



$$m\frac{dv}{dt} = -\frac{mg}{L}x$$

$$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

$$x = A\sin(\sqrt{\frac{g}{L}}t + \alpha)$$

単振動

抵抗があると

$$m\frac{dv}{dt} = -\frac{mg}{L}x + \gamma v$$

$$\frac{d^2x}{dt^2} - \gamma\frac{dx}{dt} + \frac{g}{L}x = 0$$

$x = e^{i\omega t}$ と置くと

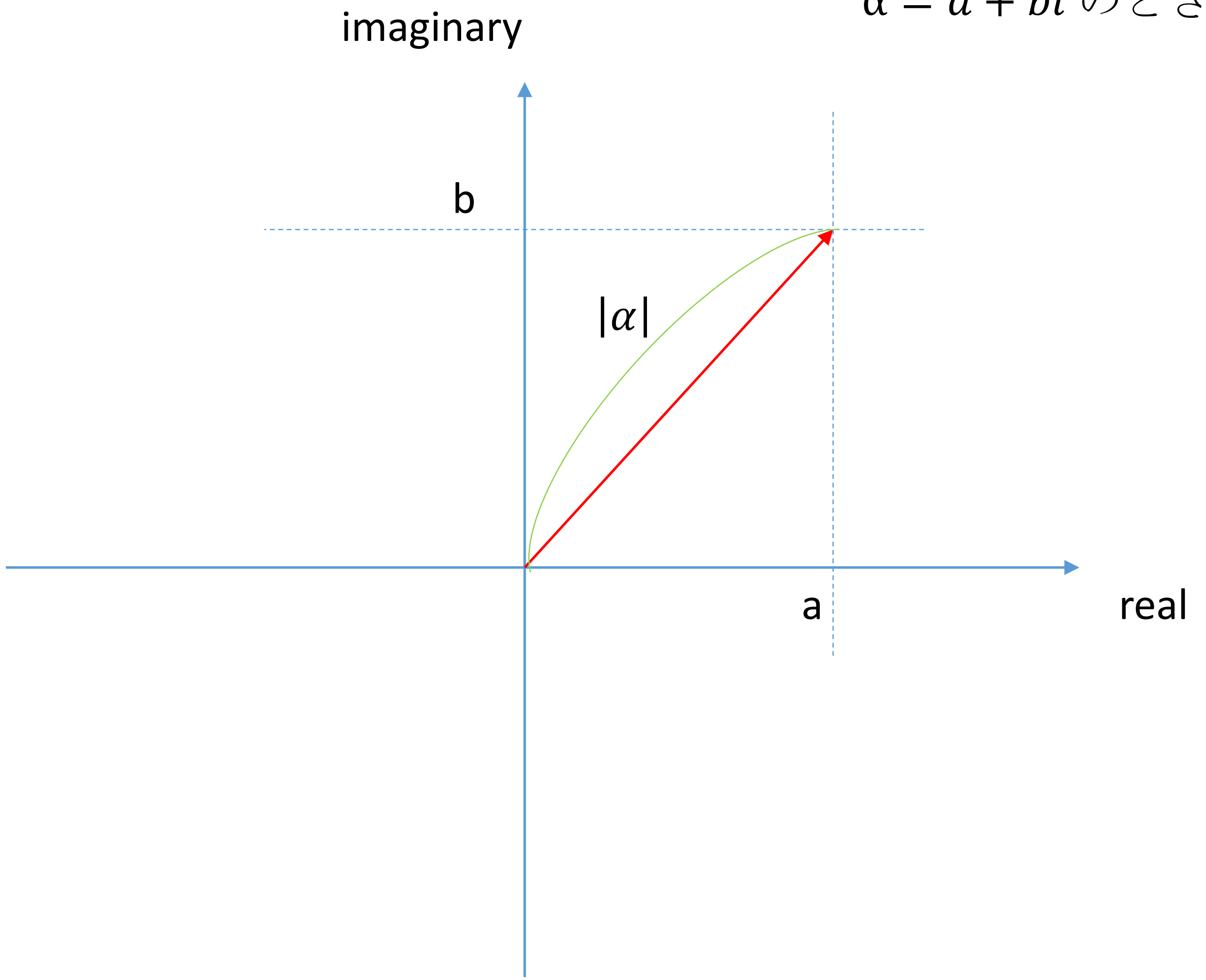
$$-\omega^2 - i\gamma\omega + \frac{g}{L} = 0$$

$$\omega = \frac{-i\gamma \pm \sqrt{-\gamma^2 + 4\frac{g}{L}}}{2} = \pm\sqrt{\frac{g}{L} - \gamma^2} - \frac{i\gamma}{2}$$

$$x = e^{-\frac{\gamma}{2}t}\exp(\pm i\sqrt{\frac{g}{L} - \gamma^2})t$$

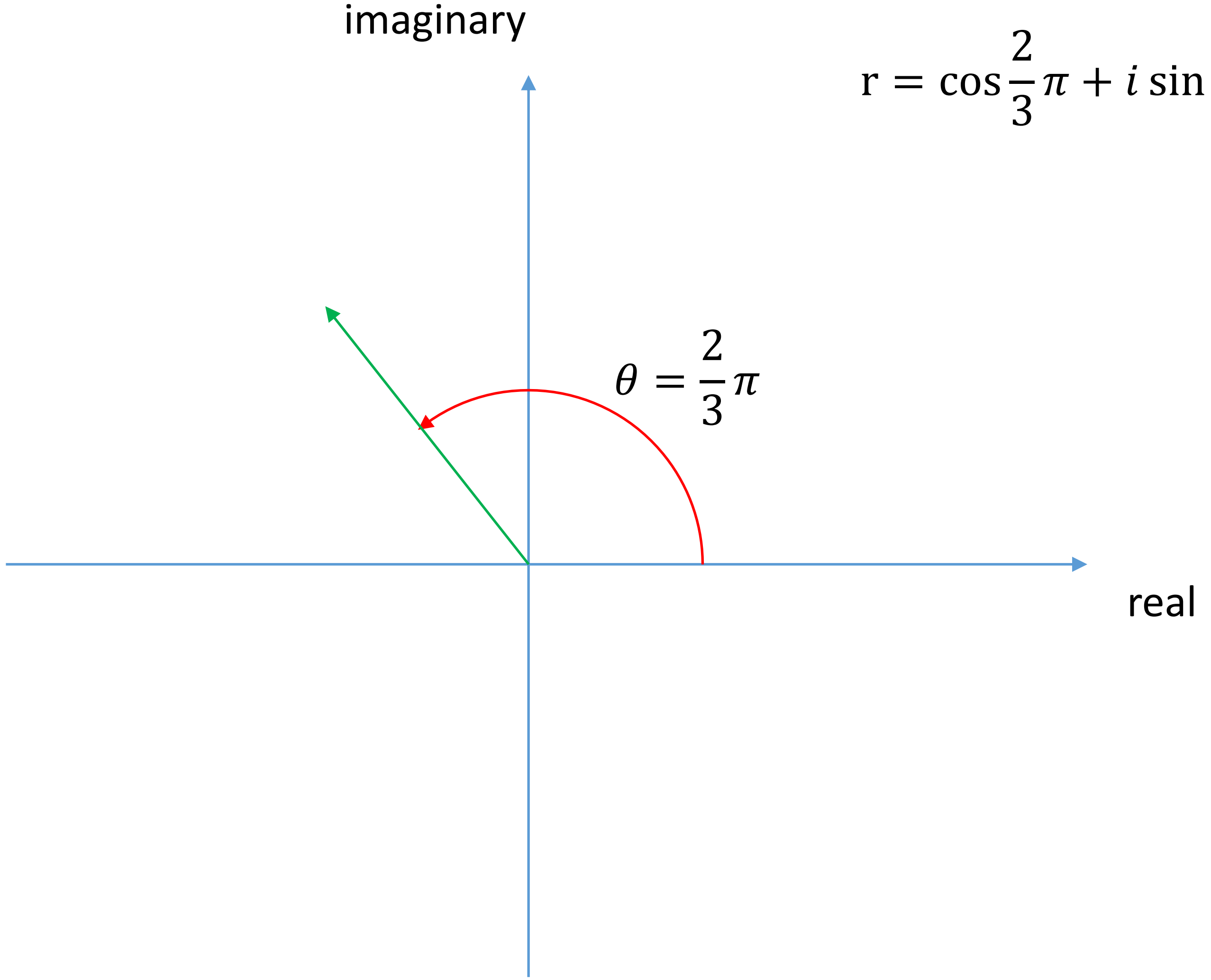
複素平面

$\alpha = a + bi$ のときの絶対値$|\alpha|$

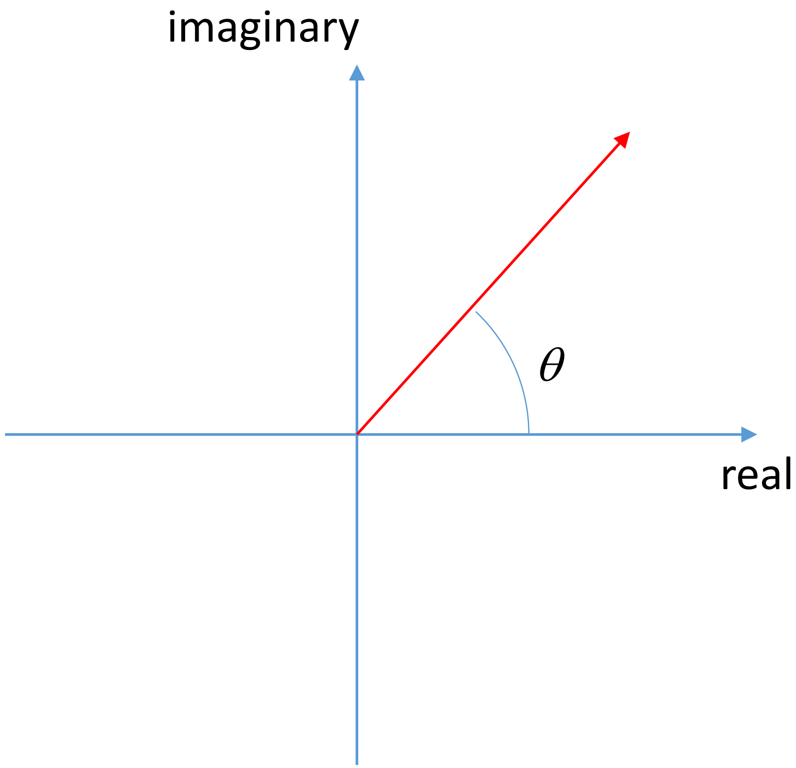


複素平面

$$r = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$$



argument



$\alpha = a + bi$ の時、$r = \sqrt{a^2 + b^2}, \theta = tan^{-1}(\frac{b}{a})$として

$a = r\cos\theta, b = r\sin\theta$ となる。

$\xi = c + di, \alpha = a + bi$ の時、$\alpha\xi$の積は

$$\alpha\xi = (a + bi)(c + di) = ac - bd + (bc + ad)i$$

$\eta = e + fi$ として、$\alpha\xi\eta$の積は？

$$\alpha\xi\eta = \big((ac - bd) + (bc + ad)i\big) * (e + fi)$$
$$= aec - bde - bcf - adf + (acf - bdf + ebc + ade)i$$

⇕

$$\alpha = r_1 e^{i\theta_1}, \xi = r_2 e^{i\theta_2}, \eta = r_3 e^{i\theta_3},$$

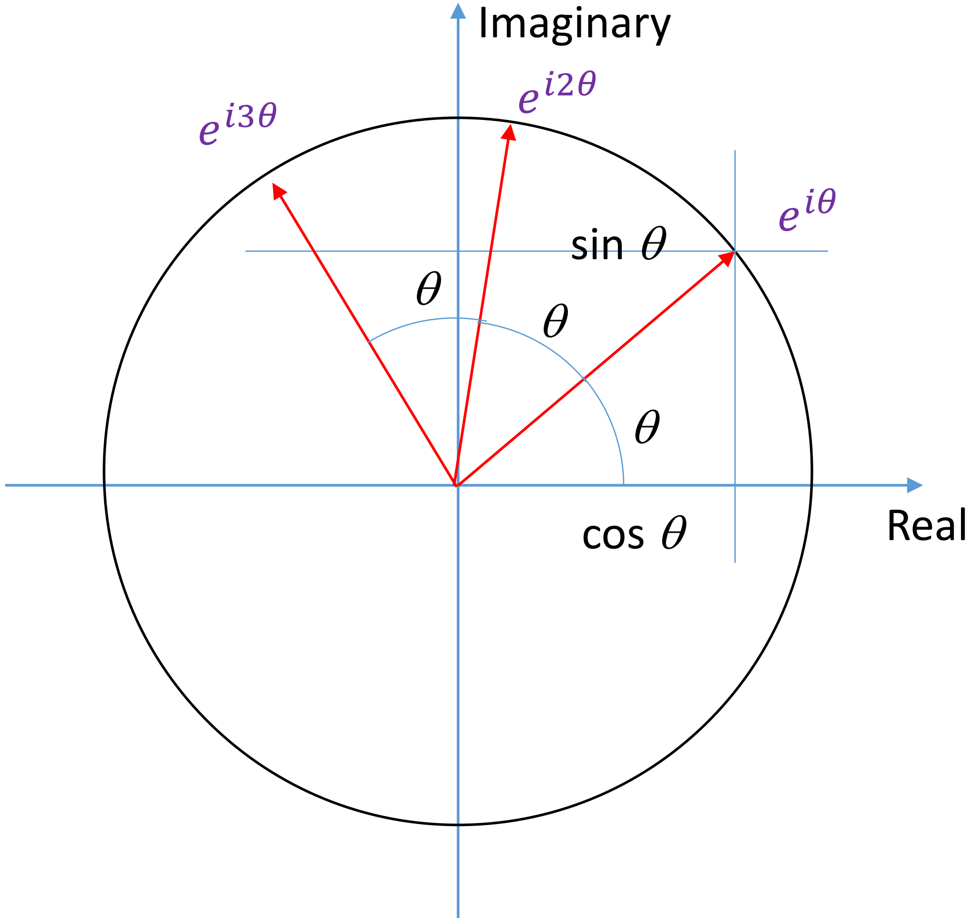
$$\alpha\xi\eta = r_1 r_2 r_3 e^{i(\theta_1 + \theta_2 + \theta_3)}$$

argument

さらに　$\frac{\alpha}{\xi\eta}$は？　　$\frac{\alpha}{\xi\eta} = \frac{a+bi}{ce-df+(de+cf)i}$,　⇔　$\alpha/\xi\eta = \frac{r_1}{r_2 r_3} e^{i(\theta_1-\theta_2-\theta_3)}$

ド・モァヴル　の公式

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$



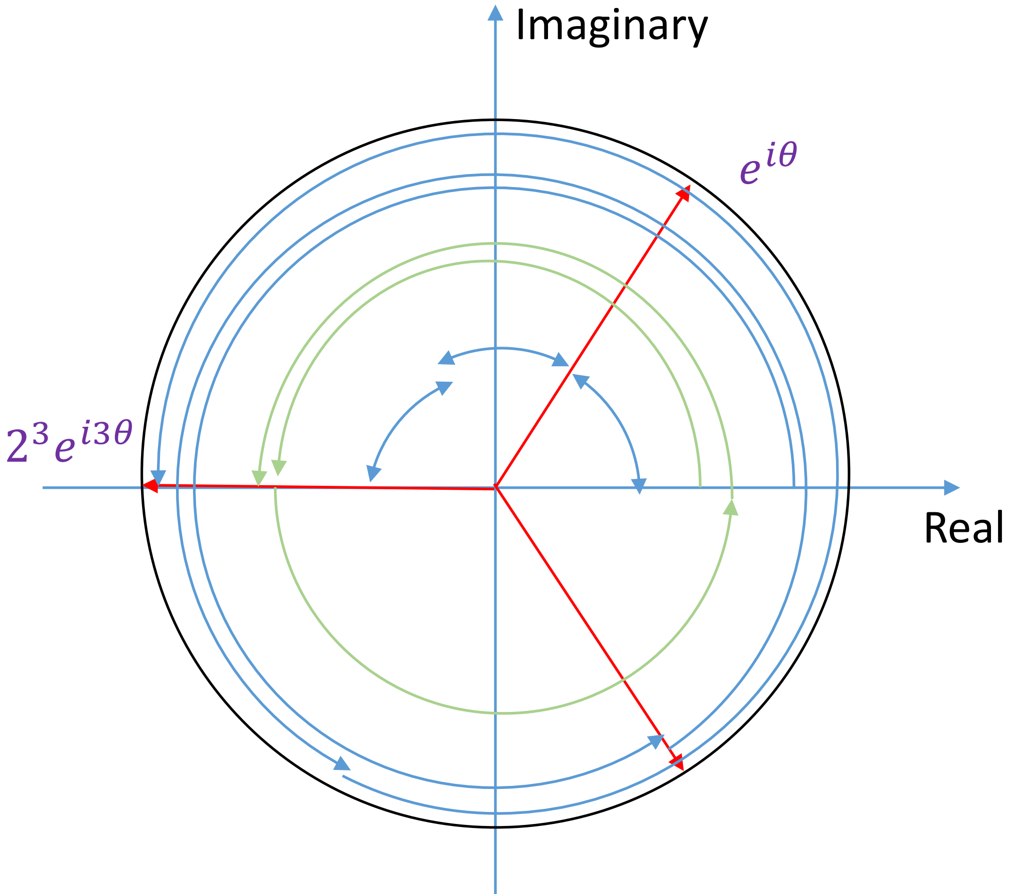
例えば

$x^3 = -8$　　の答えは？

因数分解

$$x^3 + 8 = (x+2)(x^2 + ax + b)$$
$$\Rightarrow (x+2)(x^2 - 2x + 4)$$

$$x = -2, -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3}i$$



$$\pi + 2m\pi = 3\theta$$

m=0, $\theta = \frac{\pi}{3}$

m=1, $\theta = \frac{3\pi}{3}$

m=2, $\theta = \frac{5\pi}{3}$

$$e^{\frac{\pi}{3}}, e^{\pi}, e^{\frac{5\pi}{3}}$$

$$e^{\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

三角関数の加法定理

$$\cos(a+b) = \frac{e^{i(a+b)} + e^{-i(a+b)}}{2}$$

$$= \frac{e^{ia}e^{ib} + e^{-ia}e^{-ib}}{2}$$

$$= \frac{(\cos a + i\sin a)(\cos b + i\sin b) + (\cos a - i\sin a)(\cos b - i\sin b)}{2}$$

$$= \cos a\cos b - \sin a\sin b$$

$$\sin(a+b) = \frac{e^{i(a+b)} - e^{-i(a+b)}}{2i}$$

$$= \frac{e^{ia}e^{ib} - e^{-ia}e^{-ib}}{2i}$$

$$= \frac{(\cos a + i\sin a)(\cos b + i\sin b) - (\cos a - i\sin a)(\cos b - i\sin b)}{2i}$$

$$= \sin a\cos b + \cos a\sin b$$