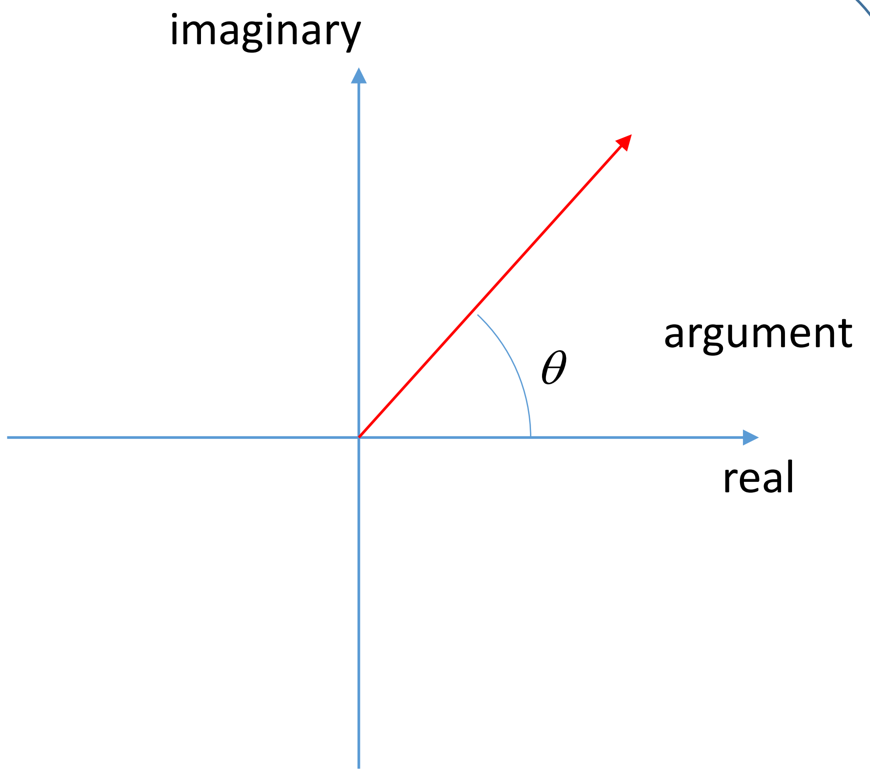
応用数学　A

先週の確認



$\alpha = a + bi$

の時、$r = \sqrt{a^2 + b^2}, \theta = tan^{-1}(\frac{b}{a})$

オイラーの公式

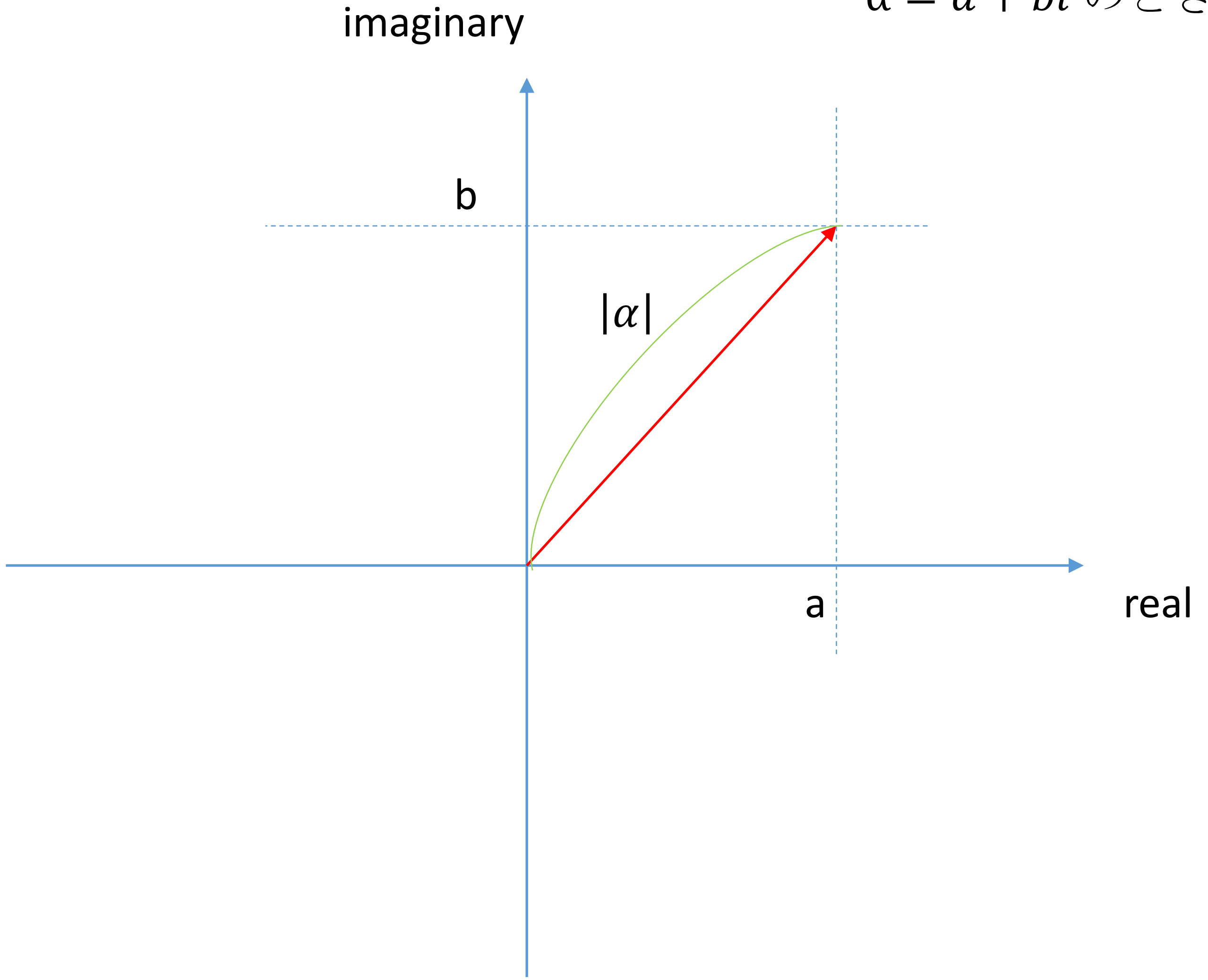
$$e^{ix} = \cos x + i \sin x$$

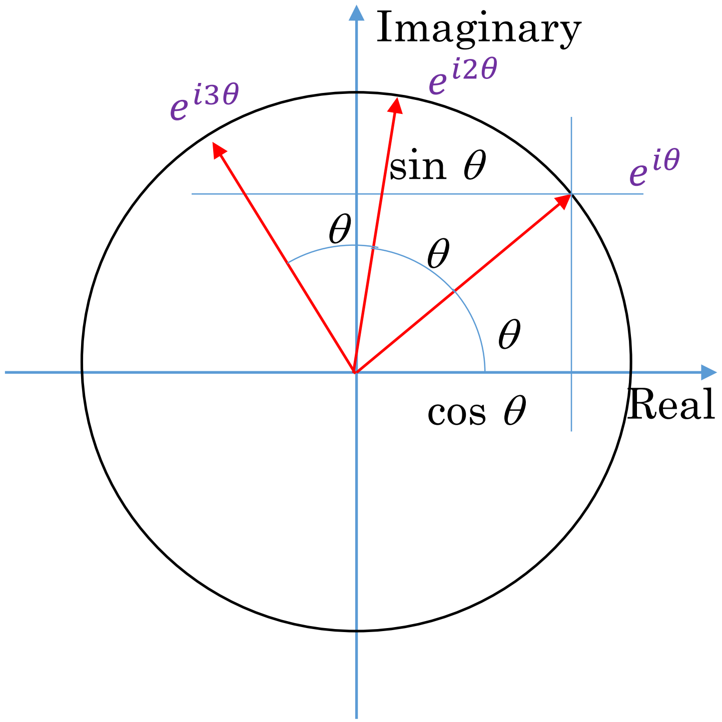
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

複素平面

$\alpha = a + bi$ のときの絶対値$|\alpha|$





例：$x^3 = 8$の答えは？

$$\left(\mathrm{r}e^{i\theta}\right)^3 = 8$$

$$r^3 e^{i3\theta} = 8$$

両辺の絶対値をとって$r^3 = 2^3$　なので、r=2

残りは、$e^{i3\theta} = 1$　これは左図でReal軸にθの回転で一致していることなので、

$$2\mathrm{m}\pi = 3\theta$$

$0 \leq \theta < 2\pi$　でこれを満たすθは、θ=0（m=0), θ=2π/3（m=1), θ=4π/3 (m=2)

したがって、

上の方程式の答えは、$\mathrm{x} = 2e^0, 2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}} = 2, -1+\sqrt{3}i, -1-\sqrt{3}i$　となる。

問題1　$x^4 = 16$の答え？

Step1　$\left(\mathrm{r}e^{i\theta}\right)^4 = r^4 e^{i4\theta} = 16$

両辺の絶対値をとって$r^4 = 2^4$　なので、$\mathrm{r} = 2$

残りは、$e^{i4\theta} = 1$　これは左図でReal軸にθの回転で一致していることなので、

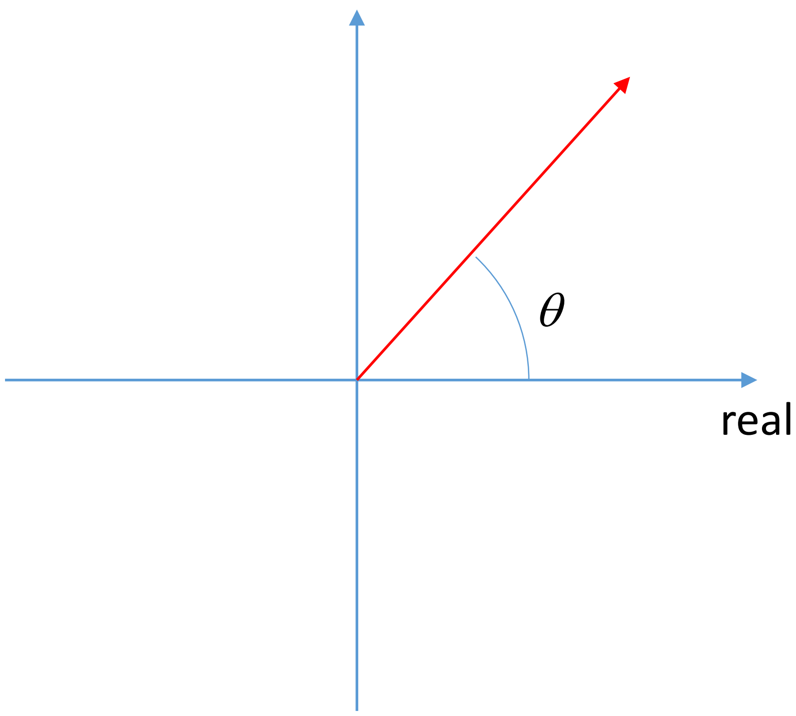
$$2\mathrm{m}\pi = 4\theta$$

Step2　θの条件は

$0 \leq \theta < 2\pi$　でこれを満たすθは、θ=0（m=0), θ=2π/4 (m=1), θ=4π/4 (m=2) , θ=6π/4 (m=3) , θ=8π/4 (m=4)

答えは、$\mathrm{x} = 2e^0, 2e^{i\frac{\pi}{2}}, 2e^{i\pi}, 2e^{i\frac{3\pi}{2}} = 2,\ \ 2i, -2, -2i$

argument



$\alpha = a + bi$　の時、$r = \sqrt{a^2 + b^2}, \theta = tan^{-1}(\frac{b}{a})$として

$a = r\cos\theta\,, b = r\sin\theta$　となる。

$\xi = c + di, \alpha = a + bi$　の時、$\alpha\xi$の積は

$$\alpha\xi = (a + bi)(c + di) = ac - bd + (bc + ad)i$$

$\eta = e + fi$　として、$\alpha\xi\eta$の積は？

$$\alpha\xi\eta = \big((ac - bd) + (bc + ad)i\big) * (e + fi)$$
$$= aec - bde - bcf - adf + (acf - bdf + ebc + ade)i$$

⇕

$$\alpha = r_1 e^{i\theta_1}, \xi = r_2 e^{i\theta_2}, \eta = r_3 e^{i\theta_3},$$

$$\alpha\xi\eta = r_1 r_2 r_3 e^{i(\theta_1 + \theta_2 + \theta_3)}$$

問題2

i　の対数　　log(i)

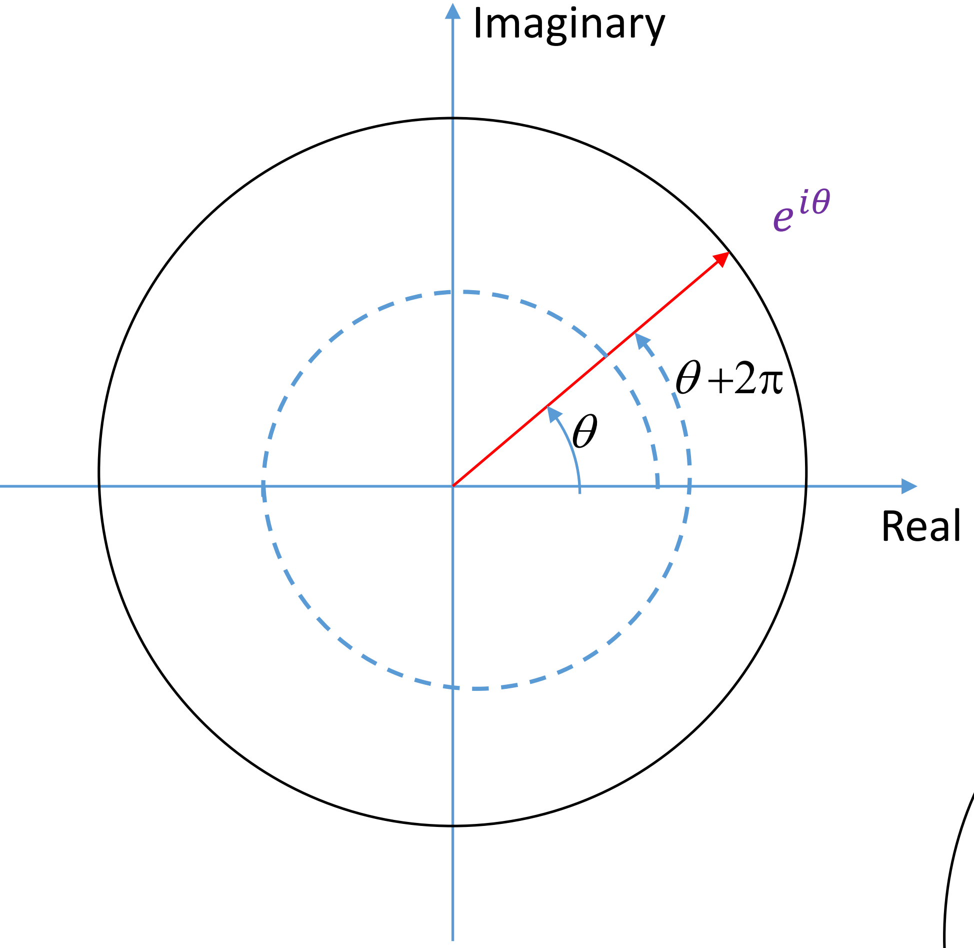
$x = \log(i)$　として両辺の指数をとる　　$e^x = i$

$e^{i\theta} = \cos\theta + i\sin\theta$　　を考えれば、右辺は、cos θ = 0, sin θ =1の状態と同じ

すなわち、　$\theta = \frac{\pi}{2}$　　$e^{i\frac{\pi}{2}} = i = e^x$

なので、$x = i\frac{\pi}{2}$　　$\log(i) = i\frac{\pi}{2}$

問題3　$\log(1+i)$?



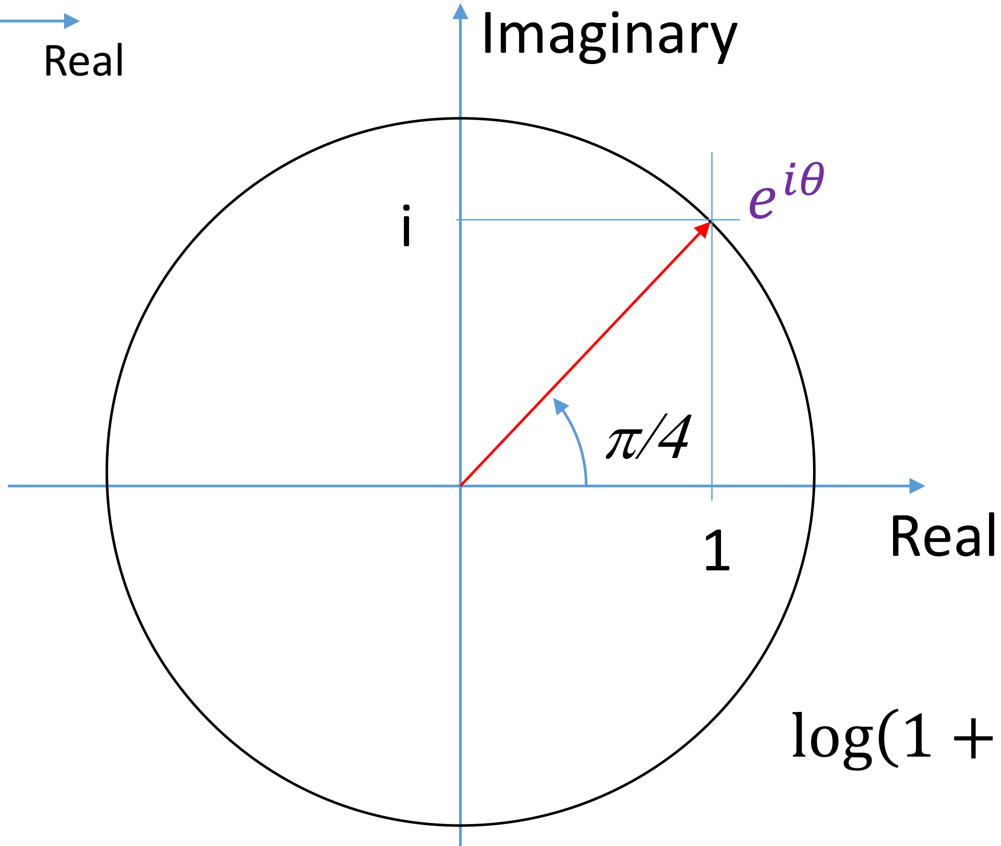
$$x = re^{i\theta} = re^{i(\theta+2m\pi)}$$

何回回っていてもargumentは同じ

$$\log(x) = \log(r) + \mathrm{i}(\theta + 2m\pi) = \log|x| + i\arg(x)$$

なので、

$$\log(1+i) = \log|1+i| + i\arg(1+i)$$



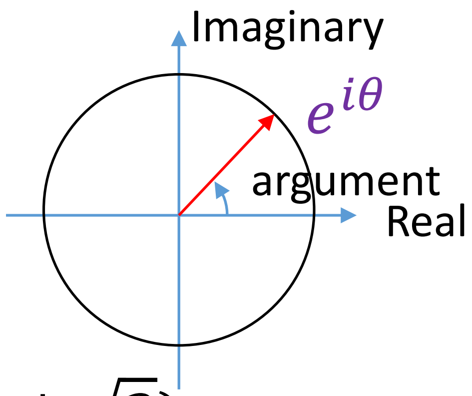
$$\log(1+i) = log\sqrt{2} + i(\frac{\pi}{4} + 2m\pi)$$

問題4 $\sin(x) = 2$

オイラーの公式

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = 2$$

$$e^{ix} - e^{-ix} - 4i = 0 \qquad y = e^{ix}$$



$$y^2 - 4iy - 1 = 0$$

$$y = 2i \pm \sqrt{-4+1} = 2i \pm i\sqrt{3} = i(2 \pm \sqrt{3})$$

$e^{ix} = y = i(2+\sqrt{3})$ として両辺の対数をとると、

$$ix = \log\left(i(2 \pm \sqrt{3})\right) = \log(2 \pm \sqrt{3}) + i(\frac{\pi}{2} + 2m\pi)$$

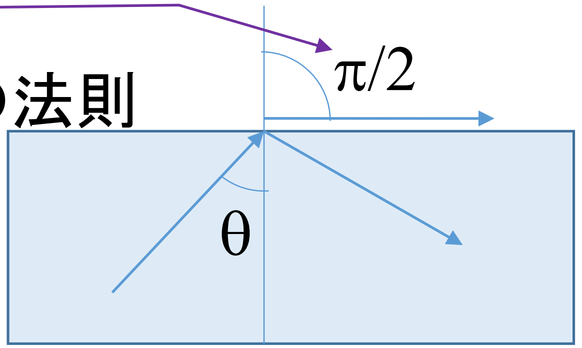
純虚数 => argument はπ/2

$$x = \frac{\pi}{2} + 2m\pi - i\log(2 \pm \sqrt{3})$$

実部は常にπ/2

$$n \sin\theta = \sin x$$

Snellの法則

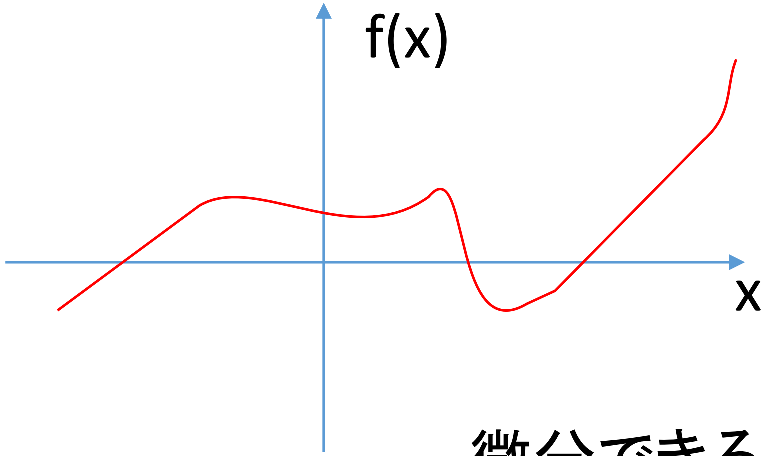


複素数の微分

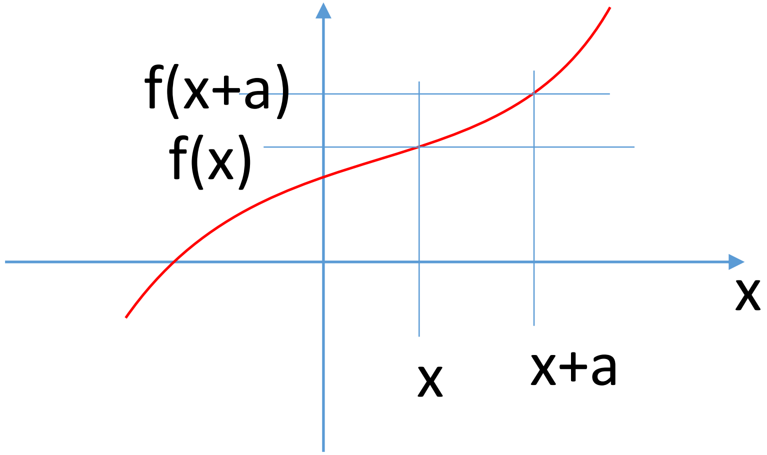
$$f'(x) = \frac{df(x)}{dx}$$

$$z = x + iy$$
$$f'(z)???$$

微分



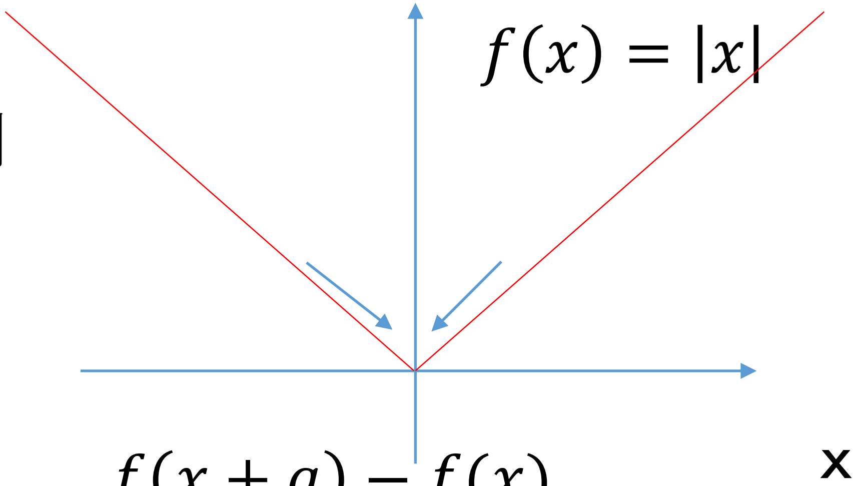
微分できる？



$$\lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

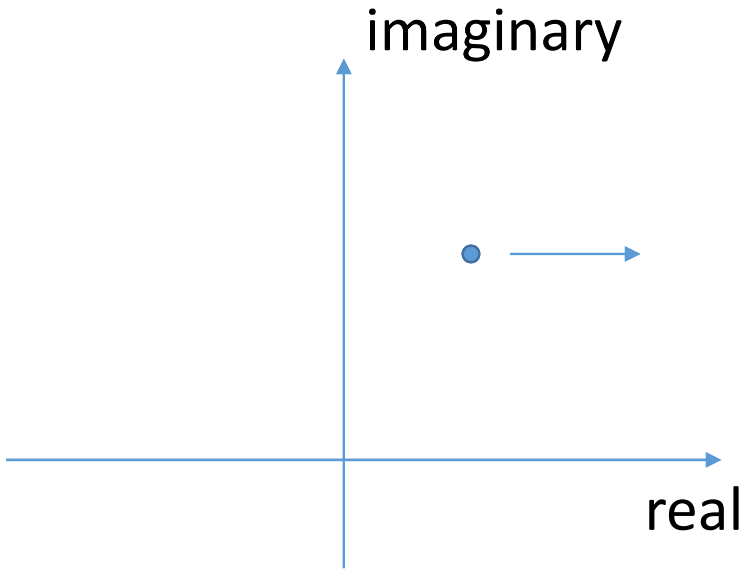
が存在するならば、微分可能

だめな代表例



$$\lim_{a \to +0} \frac{f(x+a) - f(x)}{a} \neq \lim_{a \to -0} \frac{f(x+a) - f(x)}{a}$$

複素数では？　$z = x + iy \quad w = f(z) = u(x, y) + iv(x, y)$ 複素数の関数



複素数での増減

$$\Delta w = f(z + \Delta z) - f(z) \qquad \Delta z = \Delta x + i\Delta y$$

実部の増減

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$

虚部の増減

$$\Delta v = v(x + \Delta x, y + \Delta y) - v(x, y)$$

1．変化がReal に沿った場合　$\Delta z = \Delta x$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \to \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

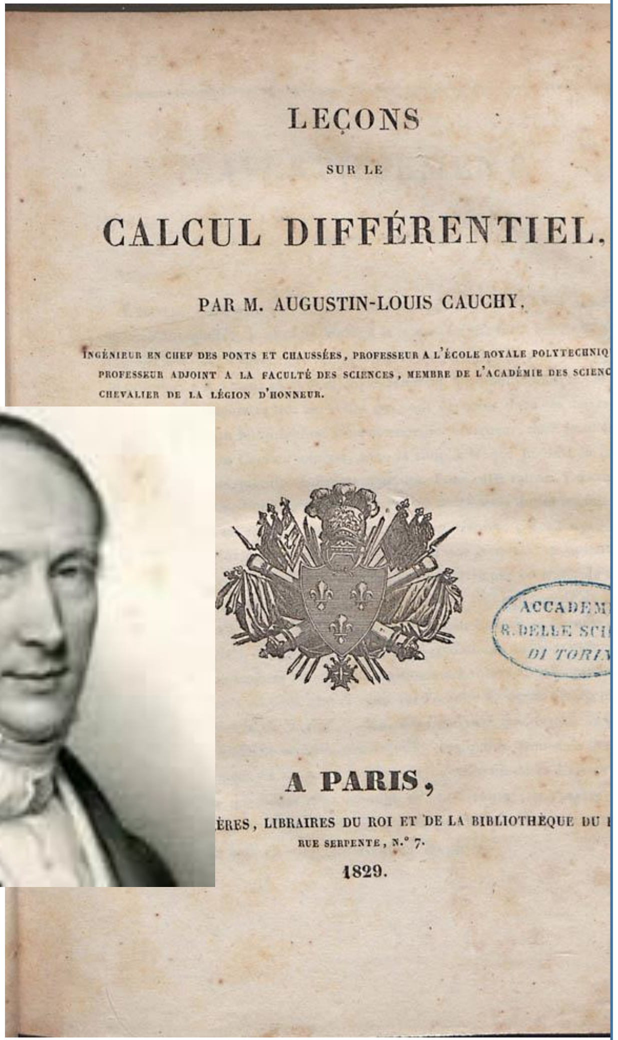
2．変化がImaginary に沿った場合　$\Delta z = \Delta y$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y} \to \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

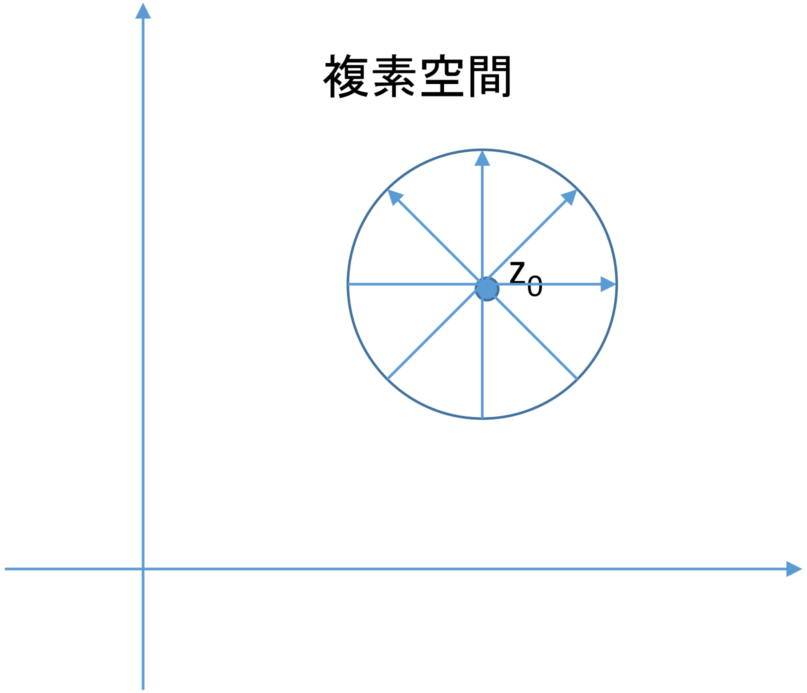
であるならば、微分可能　　正則

コーシー・リーマンの条件



Augustin Louis Cauchy
1789-1857
Ecole Polytechnic, France

- Riemann bilinear relations
- Riemann conditions
- Riemann form
- Riemann function
- Riemann–Hurwitz formula
- Riemann matrix
- Riemann operator
- Riemann singularity theorem
- Riemann surface
  - Compact Riemann surface
- The tangential Cauchy–Riemann complex
- Zariski–Riemann space
- Cauchy–Riemann equations
- Riemann integral
  - Generalized Riemann integral
  - Riemann multiple integral
- Riemann invariant
- Riemann mapping theorem
  - Measurable Riemann mapping theorem
- Riemann problem
- Riemann solver
- Riemann sphere
- Riemann–Hilbert correspondence
- Riemann–Hilbert problem
- Riemann–Lebesgue lemma
- Riemann–Liouville differintegral
- Riemann–Roch theorem
  - Arithmetic Riemann–Roch theorem
  - Riemann–Roch theorem for smoo
  - Grothendieck–Hirzebruch–Riemar
  - Hirzebruch–Riemann–Roch theore
- Riemann–Stieltjes integral
- Riemann series theorem
- Riemann sum
- Riemann–von Mangoldt formula
- Riemann hypothesis
  - Generalized Riemann hypothesis
  - Grand Riemann hypothesis
  - Riemann hypothesis for curves ov
- Riemann theta function
- Riemann Xi function
- Riemann zeta function
- Riemann–Siegel formula
- Riemann–Siegel theta function
- Free Riemann gas
- Riemann–Cartan geometry
- Riemann–Silberstein vector
- Riemann curvature tensor
- Riemann tensor (general relativity)



$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

が　z_0への近づき方によらず一定の値を持つ

微分可能

z^nは微分可能か？

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = nz^{n-1}$$

$$(z + \Delta z)^n = z^n + nz^{n-1}\Delta z + \frac{1}{2}n(n-1)z^{n-2}(\Delta z)^2$$

例：$f(z) = z^3$は複素空間 $z = x + iy$ で微分可能（正則）か？

$$w = f(z) = u(x,y) + iv(x,y) = (x+iy)^3 = \underbrace{x^3 - 3xy^2}_{u(x,y)} + i(\underbrace{3x^2y - y^3}_{v(x,y)})$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 - 3y^2 + i(6xy)$$

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - i(-6xy)$$

互いに等しい　　正則（微分可能）

例：f(z) = 1/(z − 1)は複素空間　$z = x + iy$ で微分可能（正則）か？

$$w = f(z) = u(x,y) + iv(x,y) = \frac{1}{x+iy-1} = \underbrace{\frac{x-1}{(x-1)^2+y^2}}_{u(x,y)} + i\underbrace{\frac{-y}{(x-1)^2+y^2}}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1-(2x-2)(x-1)}{((x-1)^2+y^2)^2} + i\frac{y(2x-2)}{((x-1)^2+y^2)^2}$$
$$= \frac{-2x^2+4x-1}{((x-1)^2+y^2)^2} + i\frac{2xy-2y}{((x-1)^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{-1+2y^2}{\left((x-1)^2+y^2\right)^2} - i\frac{-2y(x-1)}{\left((x-1)^2+y^2\right)^2}$$
$$= \frac{2y^2-1}{\left((x-1)^2+y^2\right)^2} + i\frac{2xy-2y}{\left((x-1)^2+y^2\right)^2}$$

等しくない
正則（微分可能）でない

コーシー・リーマンの条件が成り立つなら

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

u, v, x, y は実数　　実数項、虚数項どうしで等しい

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

例　(cos z)’を求める。　$z = x + iy$

$$\cos(x+iy) = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2} = \frac{(\cos x + i\sin x)e^{-y} + (\cos x - i\sin x)e^{y}}{2}$$

$$= \underbrace{\cos x\frac{e^{y}+e^{-y}}{2}}_{u} \underbrace{- i\sin x\frac{e^{y}-e^{-y}}{2}}_{v}$$

$$\frac{\partial u}{\partial x} = -\sin x\frac{e^{y}+e^{-y}}{2}, \qquad \frac{\partial v}{\partial y} = -\sin x\frac{e^{y}+e^{-y}}{2} -$$

$$-\frac{\partial v}{\partial x} = \cos x\frac{e^{y}-e^{-y}}{2}, \qquad \frac{\partial u}{\partial y} = \cos x\frac{e^{y}-e^{-y}}{2}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$(\cos z)' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\sin x\frac{e^{y}+e^{-y}}{2} - i\cos x\frac{e^{y}-e^{-y}}{2}$$

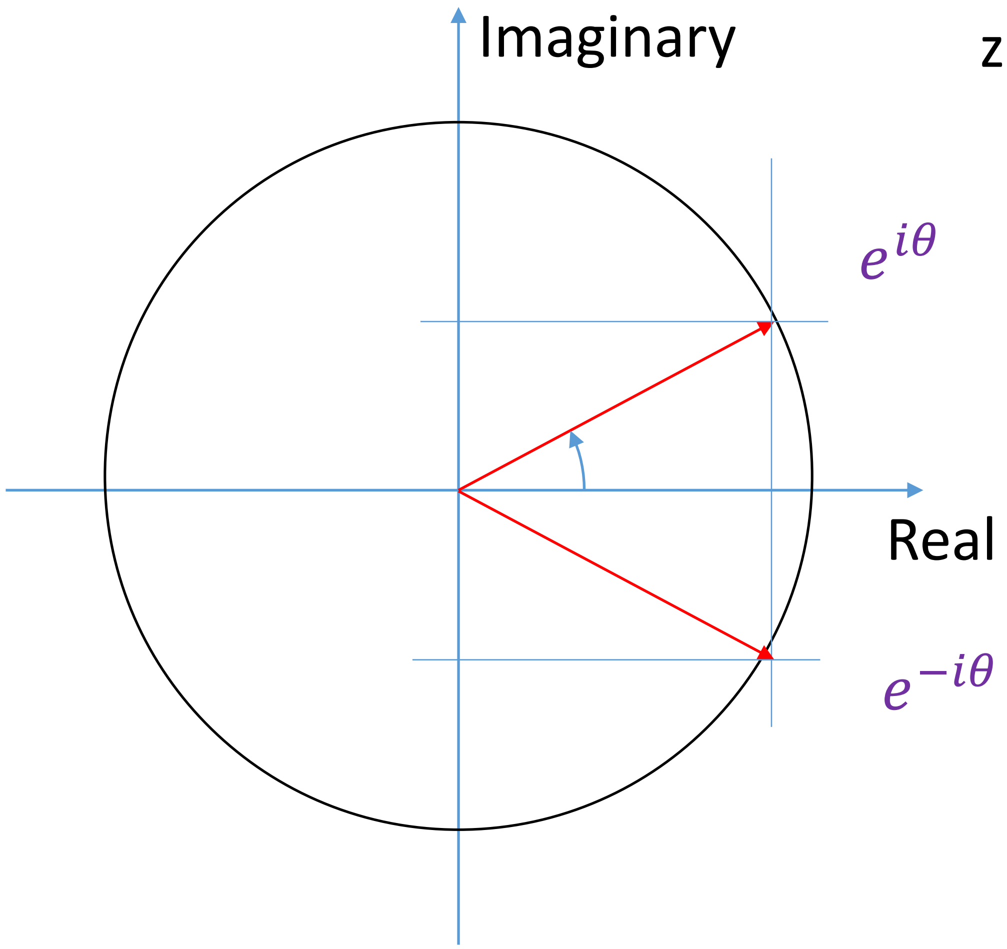
$$= -i\frac{(\cos x - i\sin x)e^{y}}{2} + i\frac{(\cos x + i\sin x)e^{-y}}{2} = i\frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2} = -\sin(x+iy)$$

複素共役と調和関数

$f(z) = 1/z$ は　　$z = x + iy = re^{i\theta}$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = r^{-1}e^{-i\theta}$$

単位円上では
z と 1/z は複素共役



コーシー・リーマンの条件　$w=f(z)=u(x,y)+iv(x,y)$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \quad \Rightarrow \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

もし、ある領域で正則であるなら、

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0$$

同様に

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=-\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y\partial x}=0$$

$$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \qquad \triangle u=0 \qquad \triangle v=0$$

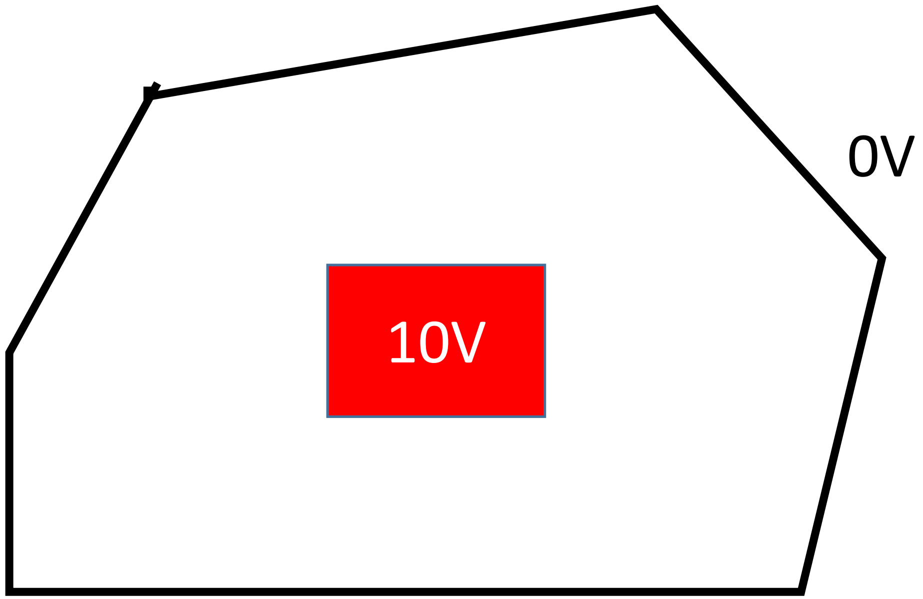
調和関数

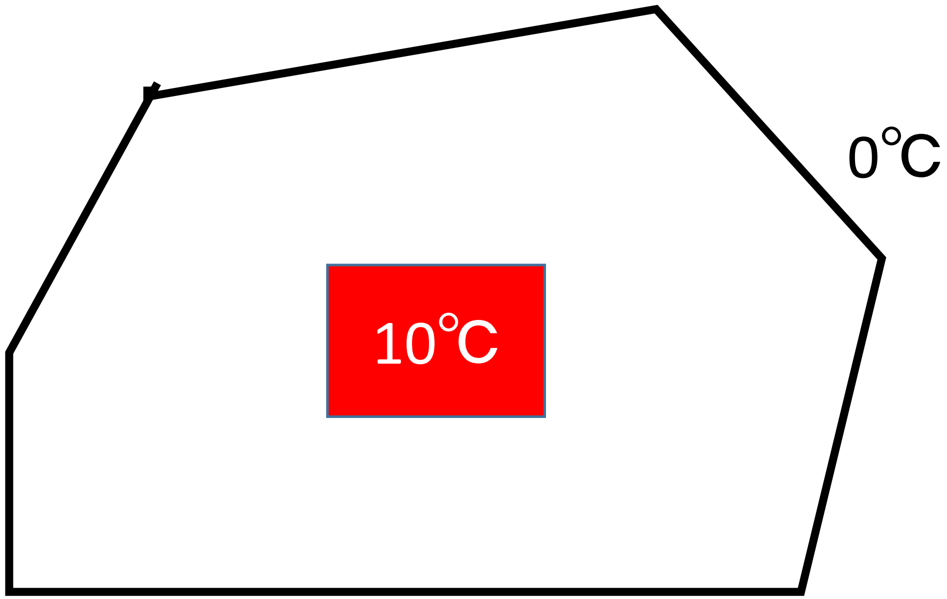
ポアソン方程式

調和関数

2次元静電場の電位　$\triangle\phi = 0$

2次元の温度分布　$\triangle T = 0$





いくつかの性質

$$\iint_S \frac{\partial\phi}{\partial n} dS = 0$$

$$\iiint_V (\phi \triangle \psi - \psi \triangle \phi) dV = \iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS = 0$$

例

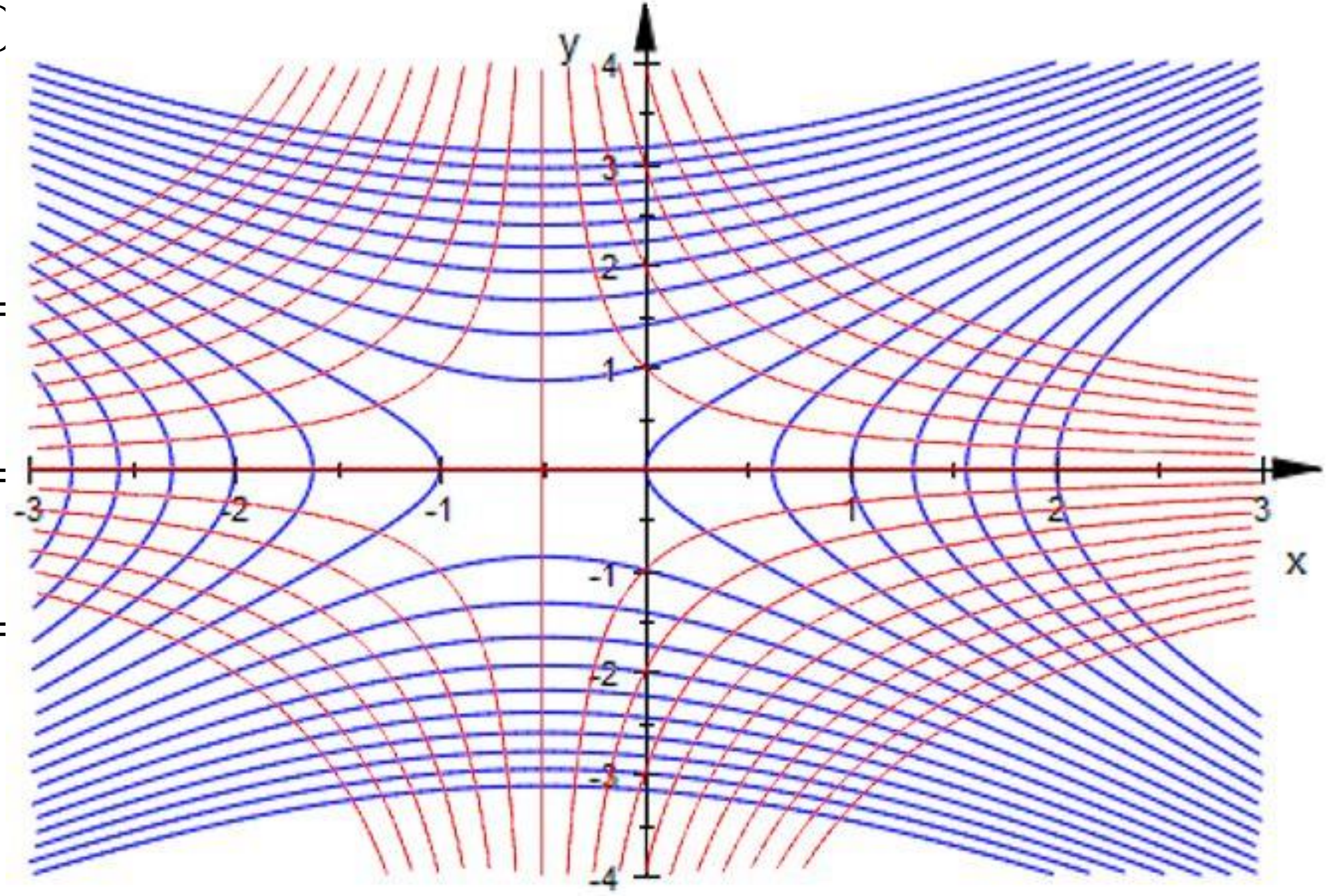
$u(x,y)$

$= 0$ 調和関数

$\frac{\partial u}{\partial x} =$

$\frac{\partial v}{\partial x} =$

$v =$



$v = 2xy + y + C$

共役調和関数

共役調和関数間は直交している