

# 電磁波工学V

電磁波の散乱問題、電磁波の計算解法  
数値シミュレーションの例

米田仁紀

# 電磁波の解を求める

基本的な考え方

- 微小電流源の塊

複雑な形状に対して

- モーメント法
- 差分近似法 (有限要素法、etc.)

# 電流分布 => 放射を求める

Green 関数の導出から

$$\nabla^2 \Psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Psi(\mathbf{r}, t)}{\partial t^2} = -g(\mathbf{r}, t) \quad \Psi(\mathbf{r}, t) = \frac{1}{4\pi} \int_{V'} \frac{[g]}{R} dv'$$

結局 Green 関数は  $G(R, \tau) = \frac{\delta(\tau - \frac{R}{c})}{4\pi R} = \frac{\delta(t - t' - \frac{R}{c})}{4\pi R}$

したがって  $\Psi = \int_{t'=-\infty}^t \int_{V'} g(\mathbf{r}', t') \frac{\delta(\tau - \frac{R}{c})}{4\pi R} dv' dt'$

$$= \frac{1}{4\pi} \int_{V'} \frac{g(\mathbf{r}', t - \frac{R}{c})}{R} dv' = \frac{1}{4\pi} \int_{V'} \frac{[g]}{R} dv'$$

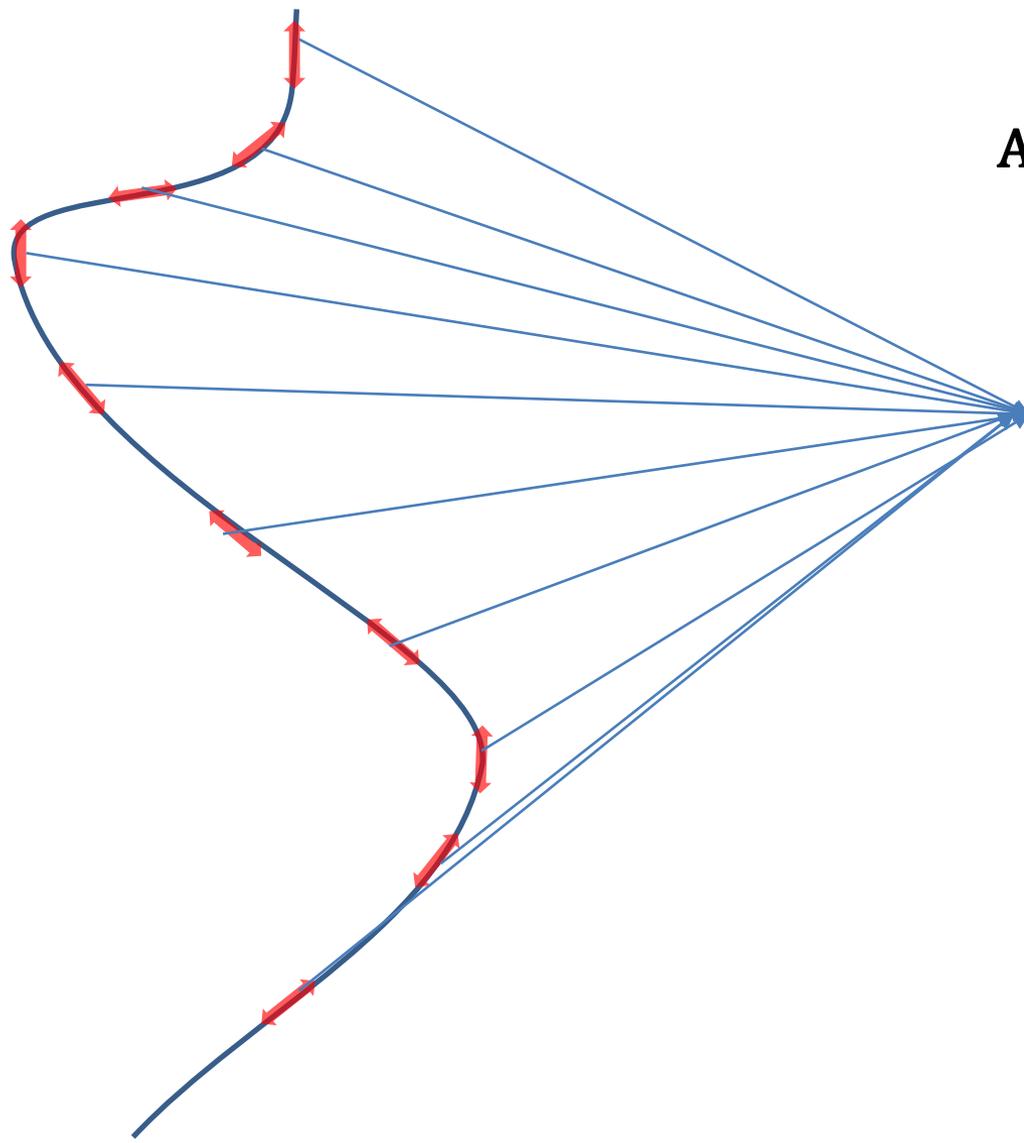
ここで  $[g] = g(\mathbf{r}', t - \frac{R}{c})$

Lorentz gaugeでは

$$\mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J$$

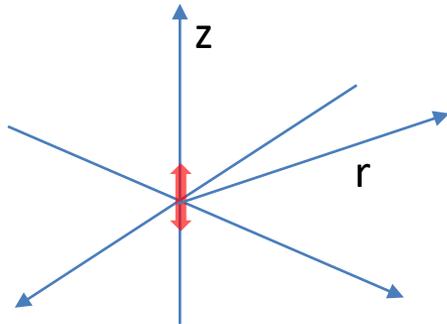
電磁波の  
ベクトルポテンシャルは

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{[\mathbf{J}]}{R} dv' \quad \leftarrow$$



$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{[\mathbf{J}]}{R} dv'$$

# Step 1 1つの微小電流源



$\mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = 0$  の解は  $e^{i(\omega t - kr)}$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{[\mathbf{J}]}{R} dv'$$

一様微小電流がz方向にlに流れている。

体積積分

$$A_z = \frac{\mu_0}{4\pi} I l e^{-ikr}$$

$$\int J(r') dv' = I l$$

Aを極座標で表すと  $A_r = A_z \cos \theta$ ,  $A_\theta = -A_z \sin \theta$ ,  $A_\phi = 0$

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} \quad \vec{E} = -j\omega(\vec{A} + \frac{1}{k^2} \nabla \nabla \cdot \vec{A})$$

$A_z$ のみ  $E_\phi = 0, H_r = 0, H_\theta = 0$

$$E_r = \frac{\eta I l}{2\pi r^2} e^{-ikr} \left(1 + \frac{1}{ikr}\right) \cos \theta$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

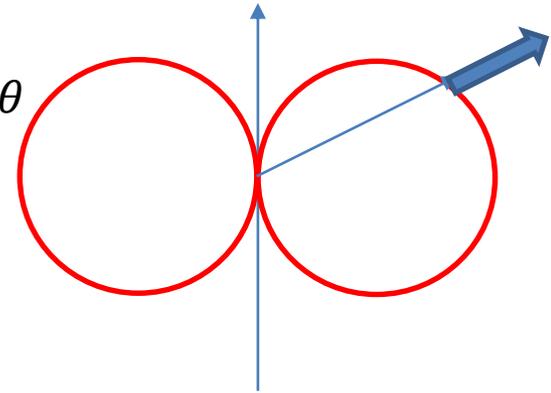
$$E_\theta = i \frac{\eta I l}{2\lambda r} e^{-ikr} \left(1 + \frac{1}{ikr} - \frac{1}{(kr)^2}\right) \sin \theta$$

$E_\theta, H_\phi$  のみ 同位相、直交

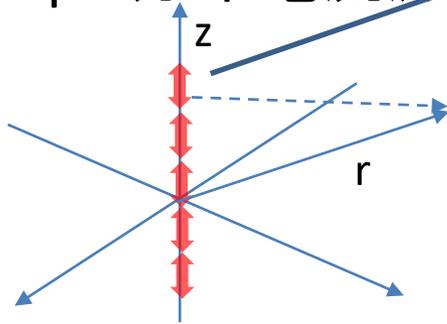
$$H_\phi = i \frac{I l}{2\lambda r} e^{-ikr} \left(1 + \frac{1}{ikr}\right) \sin \theta$$

ポインティングベクトルは？

$$\vec{S} = \frac{1}{2} \text{Re}[\vec{E} \times \vec{H}^*] = \vec{e}_r \frac{1}{2} E_\theta H_\phi = \vec{e}_r \frac{\eta (I\ell)^2}{8\lambda^2} \frac{1}{r^2} \sin^2 \theta$$



Step 2 分布電流源



全部微小なものを足す

遠方では

$$E_\theta = \sum i \frac{\eta I \ell}{2\lambda r} e^{-ikr} \sin \theta$$

$$E_\theta = i \frac{\eta}{2\lambda} \int \frac{I(z) dz}{|r - r'|} e^{-ik|r - r'|} \sin \theta$$

位相の項はkがかかっているので変化が大きい

$$\frac{1}{|r - r'|} \sim \frac{1}{r},$$

$$k|r - r'| \sim kr - kz \cos \theta$$

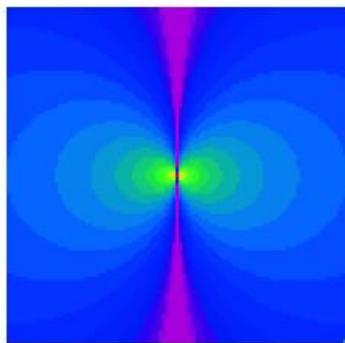
$$k = \frac{2\pi}{\lambda}$$

光ではλが小さい

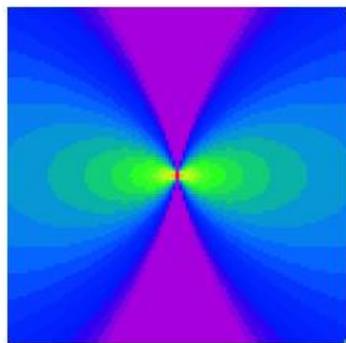
$$E_\theta = i \frac{\eta \sin \theta}{2\lambda r} e^{-ikr} \int I(z) e^{ikz \cos \theta} dz$$

$$e^{ikz \cos \theta}$$

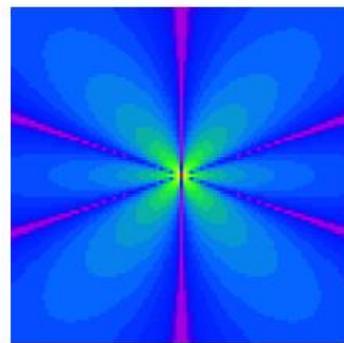
位相の項はkがかかっているので変化が大きい



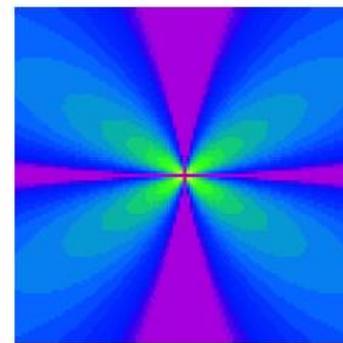
$$L = \frac{\lambda}{2}$$



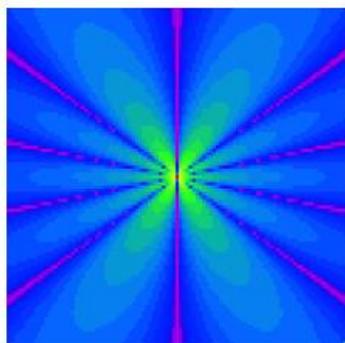
$$L = \lambda$$



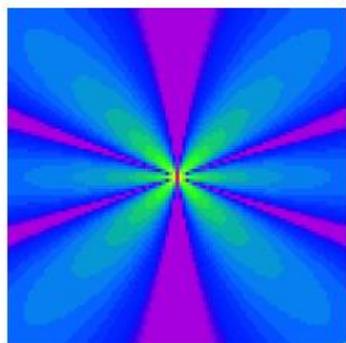
$$L = \frac{3\lambda}{2}$$



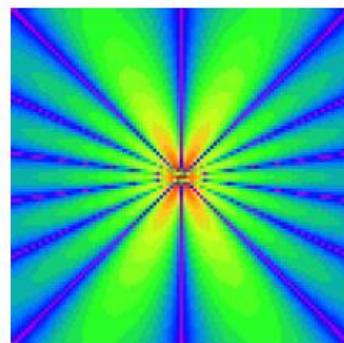
$$L = 2\lambda$$



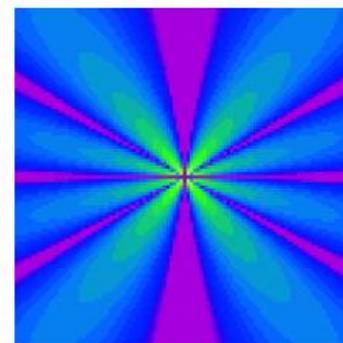
$$L = \frac{5\lambda}{2}$$



$$L = 3\lambda$$



$$L = \frac{7\lambda}{2}$$



$$L = 4\lambda$$

図2.7 アンテナ長と放射パターン

<http://www.wave.ie.niigata-u.ac.jp/yamaguchi/education/waveinformation/>

# 具体的な計算法について

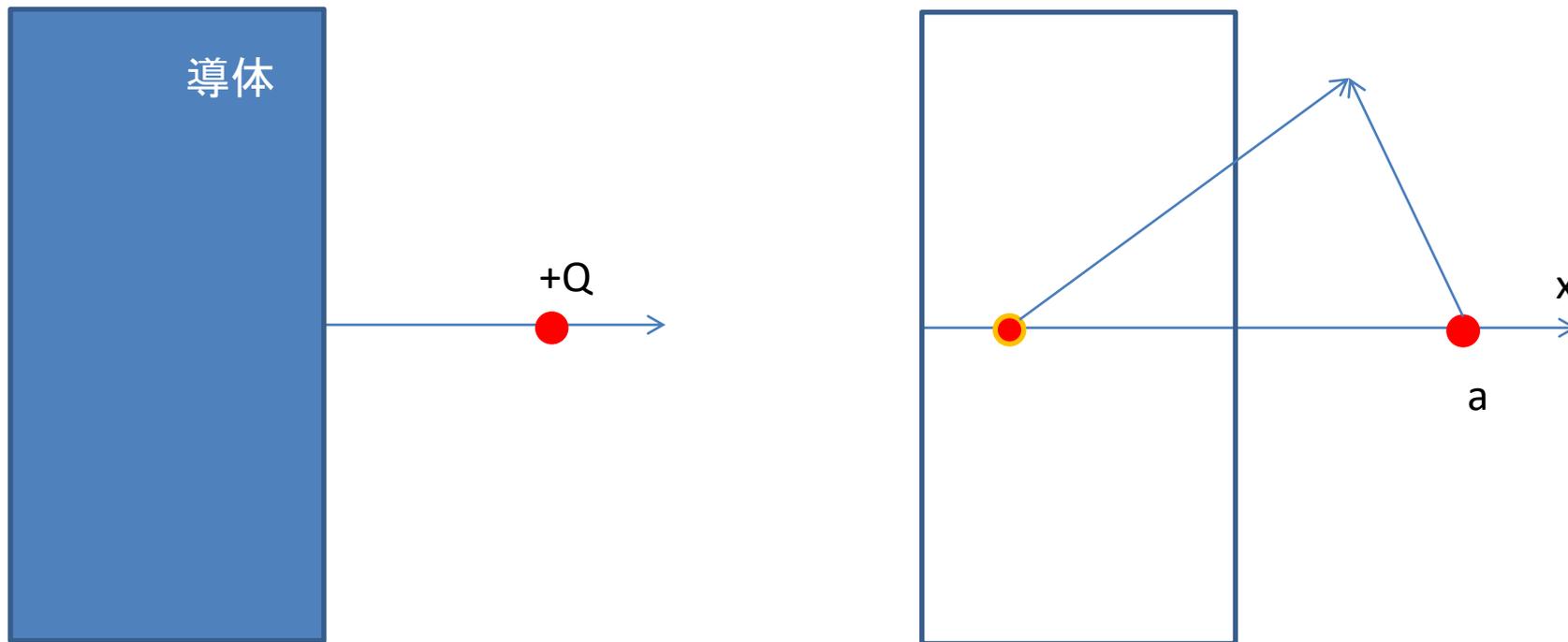
電磁波放射

電磁波散乱

モーメント法(境界要素法)

# 境界値問題： 境界要素で解が求まる

例えば、電気鏡像法もその一つ



$$U = \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$
$$= \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right]$$

電気鏡像法

$$U = \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$
$$= \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right]$$

$E = -\nabla U$ なので

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{Q}{4\pi\epsilon_0 r_1^3} \begin{pmatrix} x-a \\ y \\ z \end{pmatrix} - \frac{Q}{4\pi\epsilon_0 r_2^3} \begin{pmatrix} x+a \\ y \\ z \end{pmatrix}$$

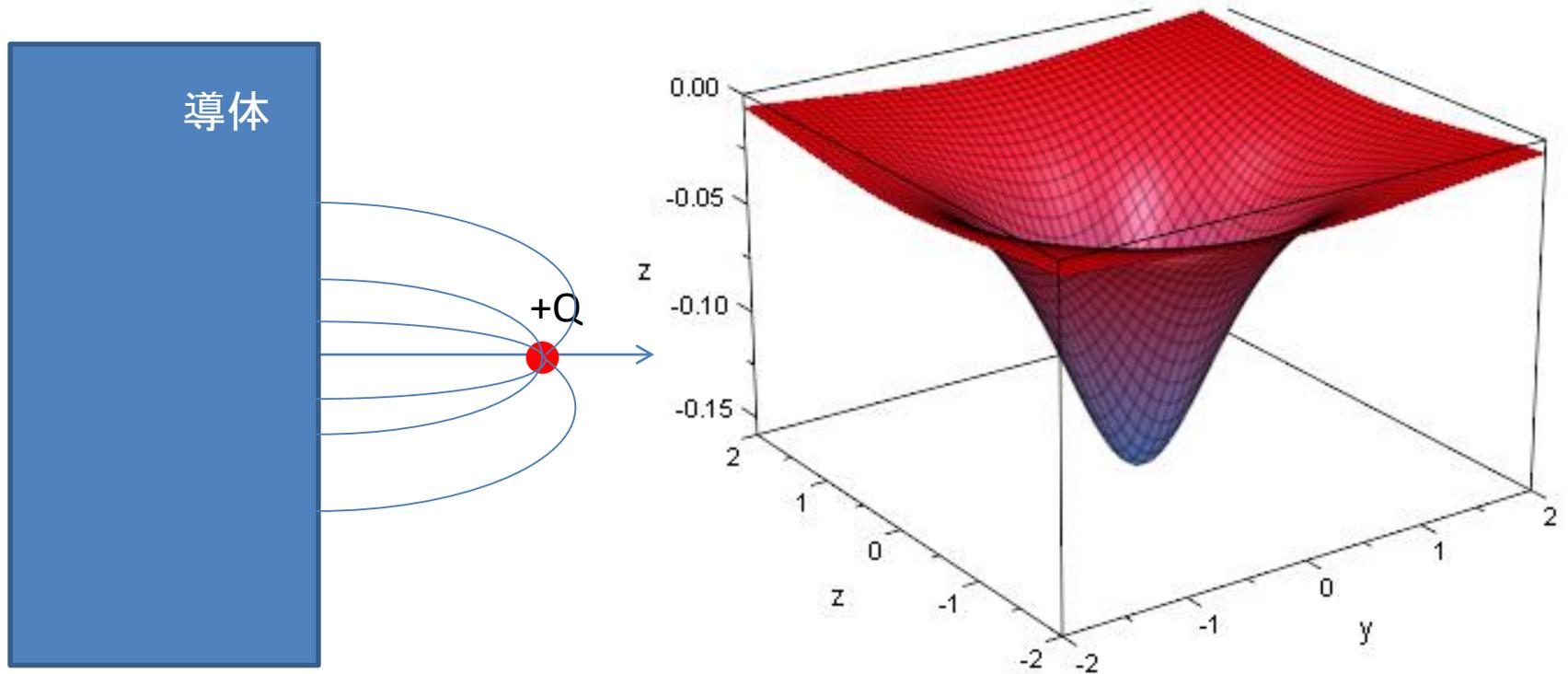
Y軸上の電界は $z=0$

$$\mathbf{E} = -\mathbf{e}_x \frac{2Q}{4\pi\epsilon_0 (a^2 + y^2)^{3/2}}$$

電荷は $\nabla \cdot \mathbf{D} = \rho$ なので  $\frac{\partial E_x}{\partial x}$  のみ

$$\sigma = \rho \Delta x = \epsilon_0 E_x(x + \Delta x) = \frac{-Qa}{2\pi(a^2 + y^2 + z^2)^{3/2}}$$

# 電気鏡像法の解の例



導体が無限なら解が求まる。  
電荷はいくつあっても重ね合わせで解ける。

# 等価波源

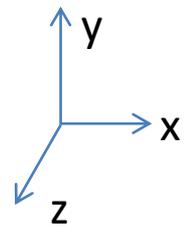
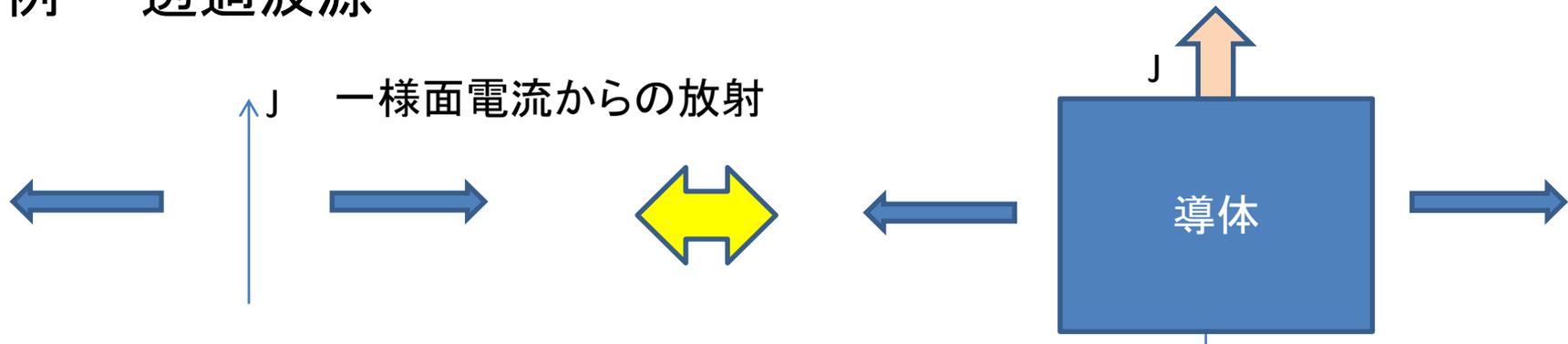
$$\begin{aligned} \phi(\mathbf{r}_o) &= \iiint_{\Omega} \frac{\rho(\mathbf{r}_s)}{\varepsilon} G(\mathbf{r}_o; \mathbf{r}_s) dv_s + \iint_{\partial\Omega} \left\{ \frac{G(\mathbf{r}_o; \mathbf{r}_s) \frac{\partial \phi(\mathbf{r}_s)}{\partial n}}{\text{電気一重層}} - \frac{\phi(\mathbf{r}_s) \frac{\partial G(\mathbf{r}_o; \mathbf{r}_s)}{\partial n}}{\text{電気二重層}} \right\} dS_s \\ &= \iiint_{\Omega} \frac{\rho(\mathbf{r}_s)}{\varepsilon} G(\mathbf{r}_o; \mathbf{r}_s) dv_s + \iint_{\partial\Omega} \left\{ \frac{G(\mathbf{r}_o; \mathbf{r}_s) (\nabla_s \phi(\mathbf{r}_s) \cdot \hat{\mathbf{n}})}{\text{電気一重層}} - \frac{\phi(\mathbf{r}_s) (\nabla_s G(\mathbf{r}_o; \mathbf{r}_s) \cdot \hat{\mathbf{n}})}{\text{電気二重層}} \right\} dS_s \end{aligned}$$

よって、

$$\begin{aligned} \mathbf{E}(\mathbf{r}_o) &= \iiint_{\Omega} \left\{ -j\omega\mu \mathbf{J}_e \psi - \mathbf{J}_m \times \nabla \psi + \frac{\rho_e}{\varepsilon} \nabla \psi \right\} dv \\ &+ \iint_{\partial\Omega} \left[ -j\omega\mu (\hat{\mathbf{n}}' \times \mathbf{H}) \psi - (\mathbf{E} \times \hat{\mathbf{n}}') \times \nabla \psi + (\mathbf{E} \cdot \hat{\mathbf{n}}') \nabla \psi \right] dS \end{aligned}$$

$$\begin{aligned} \mathbf{H}(\mathbf{r}_o) &= \iiint_{\Omega} \left\{ -j\omega\varepsilon \mathbf{J}_m \psi + \mathbf{J}_e \times \nabla \psi + \frac{\rho_m}{\mu} \nabla \psi \right\} dv \\ &+ \iint_{\partial\Omega} \left[ -j\omega\varepsilon (\mathbf{E} \times \hat{\mathbf{n}}') \psi + (\hat{\mathbf{n}}' \times \mathbf{H}) \times \nabla \psi + (\mathbf{H} \cdot \hat{\mathbf{n}}') \nabla \psi \right] dS \end{aligned}$$

# 例 透過波源



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J$$

$A_z = e^{\lambda x}$  として齊次方程式を作る

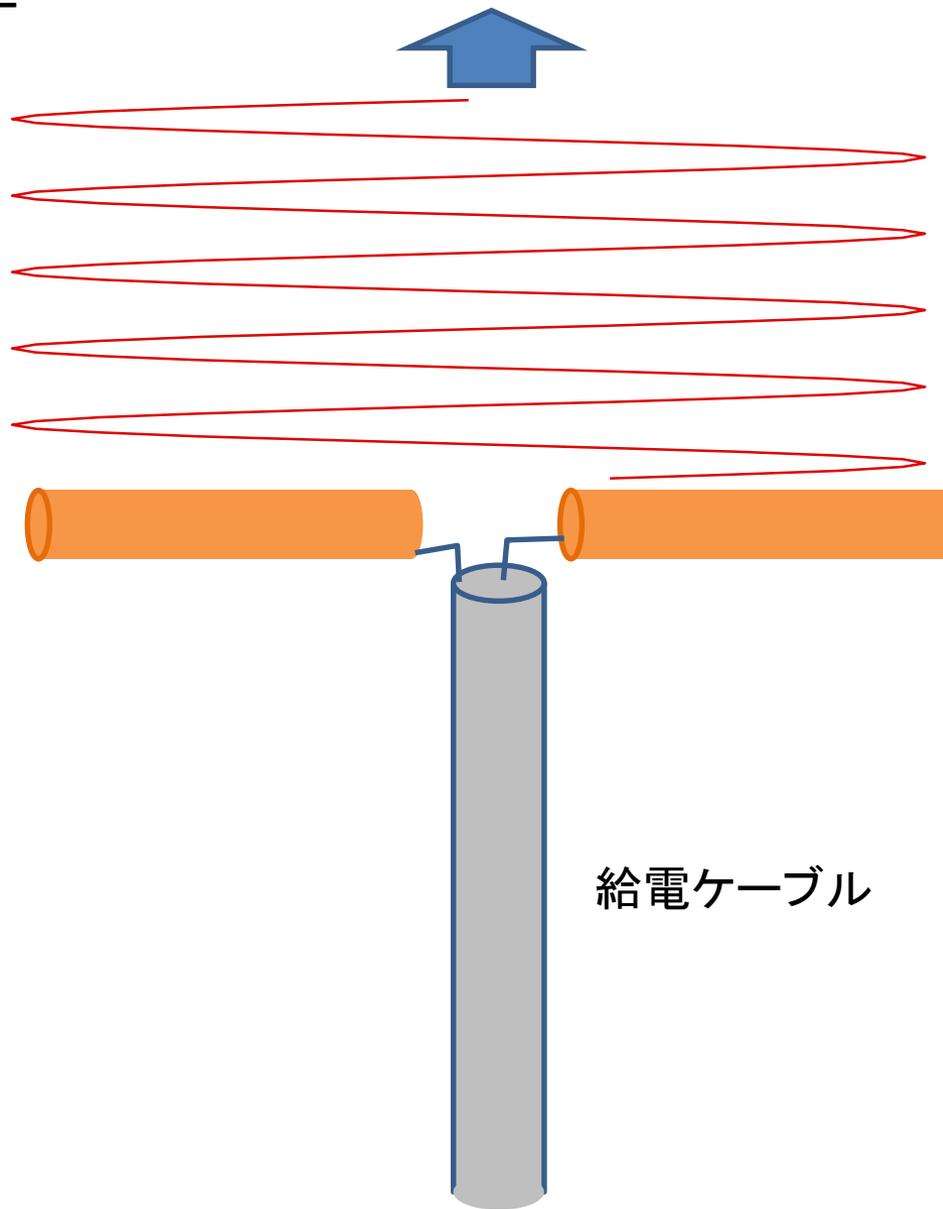
$$(\lambda^2 + k^2)e^{\lambda x} = 0 \quad \lambda = \pm ik$$

$$A_z = \begin{cases} Ae^{ikx} & x < 0 \\ Be^{-ikx} & x > 0 \end{cases}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = -\mathbf{e}_y \frac{1}{\mu} \frac{\partial A_z}{\partial x} = \begin{cases} -A \frac{ik}{\mu} e^{ikx} & x < 0 \\ B \frac{ik}{\mu} e^{-ikx} & x > 0 \end{cases} \mathbf{e}_z$$

$$\mathbf{E} = \frac{1}{i\omega \epsilon} \nabla \times \mathbf{H} = \begin{cases} -A i\omega e^{ikx} & \\ -B i\omega e^{-ikx} & \end{cases} \mathbf{e}_y$$

ダイポールアンテナ



給電ケーブル

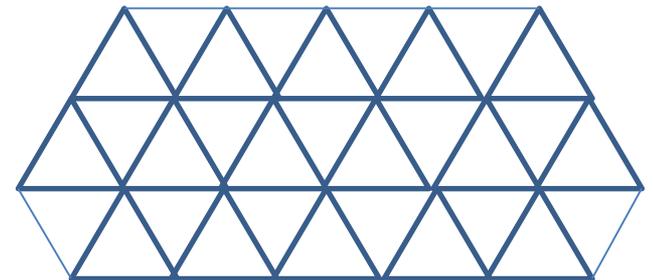
各場所での電流分布を知る必要がある

# 数値計算のモデル

- 有限要素法
- モーメント法



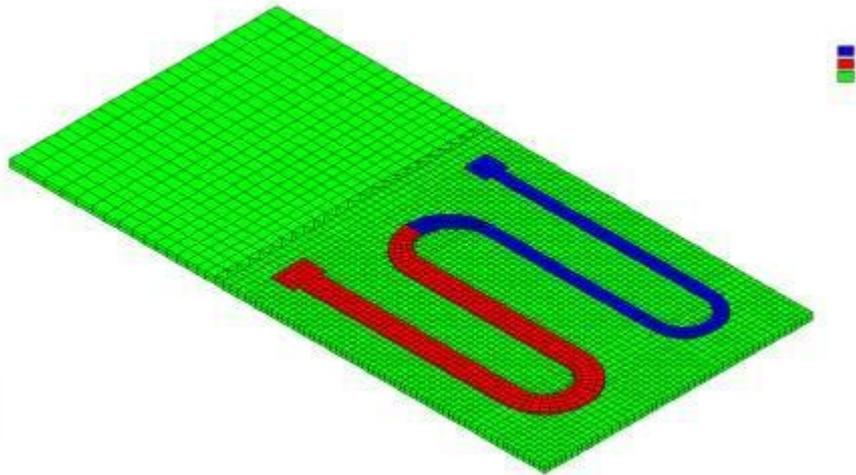
この境界で定義されたもので解く



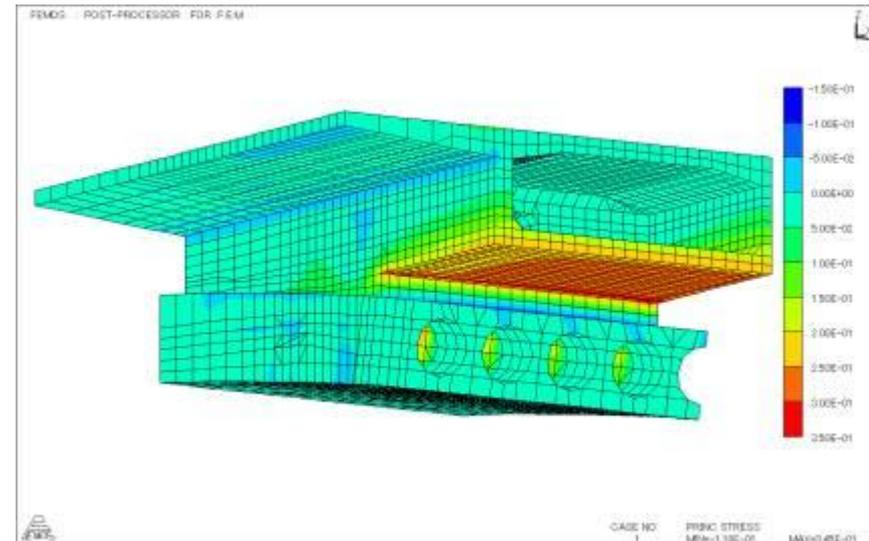
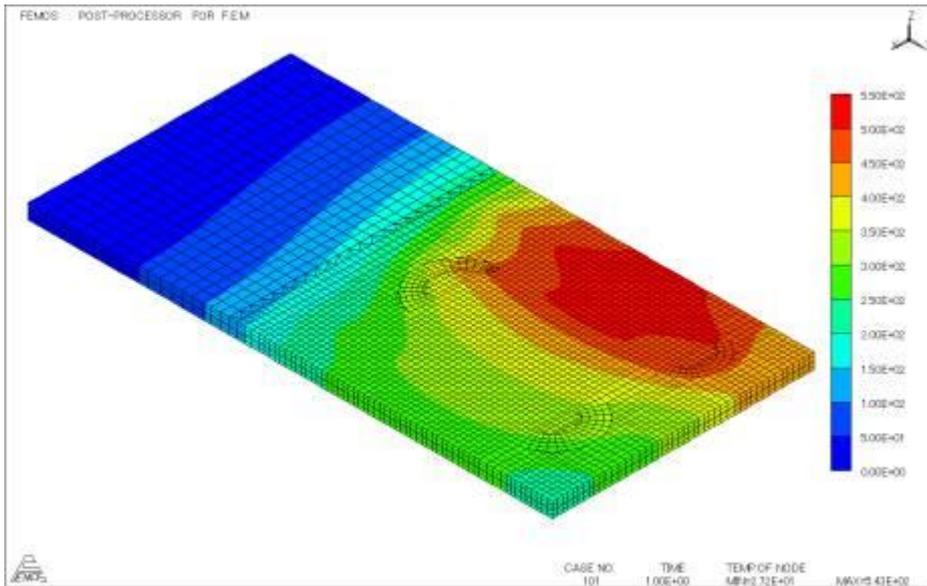
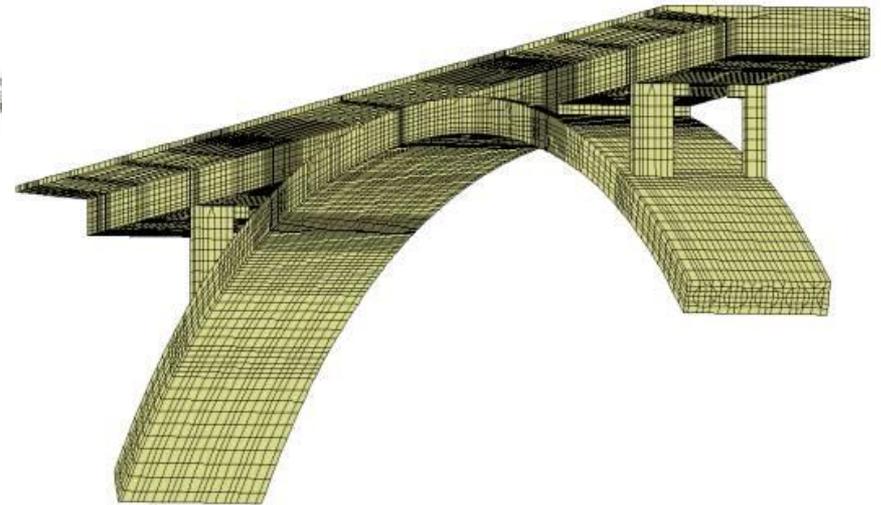
この差分点で計算をおこなう

# 有限要素法の例

## 発熱体の非定常熱伝導解析



## アーチクラウン接合部付近応力照査



# 微分方程式を差分化する

差分近似の例

$$y'' - y = -x, \quad y(0) = y(1) = 0$$

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

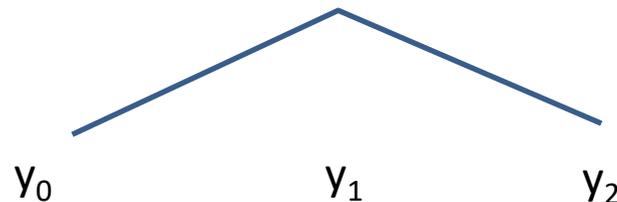
$$y'' = \frac{1}{h^2} \delta^2 y \quad (x = kh, L = Nh) \quad y' = \frac{1}{h} \delta y$$

$$y_{k-1} - (2 + h^2)y_k + y_{k+1} = -kh^2$$

$$h = \frac{1}{2} \quad \text{とすると} \quad y_0 = y_2 = 0 \text{を考えて}$$

$$\left(2 + \left(\frac{1}{2}\right)^2\right)y_1 = \left(\frac{1}{2}\right)^3$$

$$y_1 = \frac{1}{18} = 0.05556$$



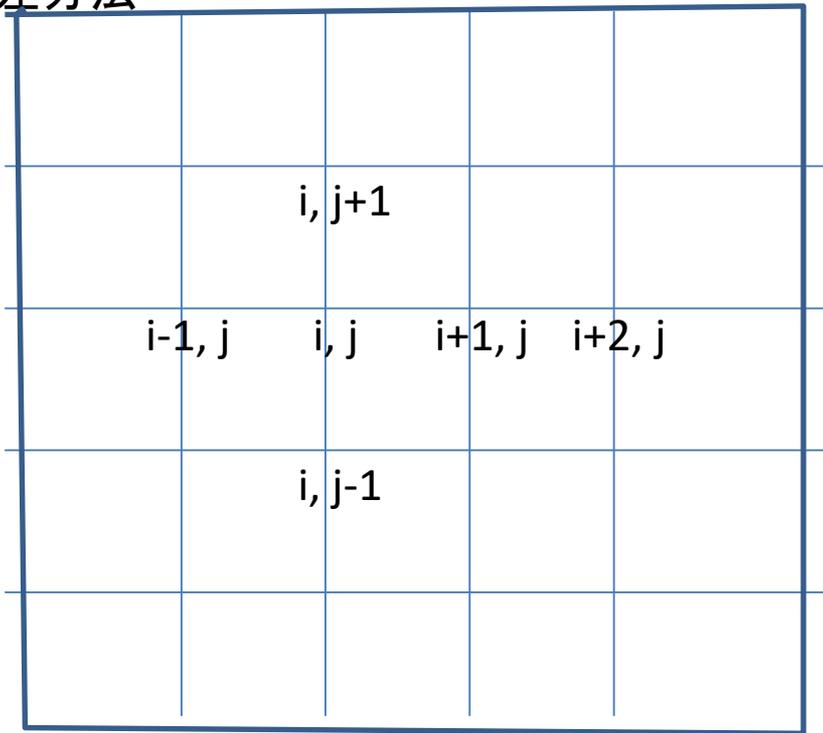
$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{d^2f(x)}{dx^2} = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(f(x + 2\Delta x) - f(x + \Delta x))/\Delta x - (f(x + \Delta x) - f(x))/\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)}{\Delta x^2}$$

# 差分法



$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= \frac{f(x(i+1), y(j)) - f(x(i), y(j))}{\Delta x} \\ &= \frac{f_{i+1, j} - f_{i, j}}{\Delta x} \end{aligned}$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{f_{i, j+1} - f_{i, j}}{\Delta y}$$

$$\frac{\partial^2 f(x, y)}{\partial x^2} = \frac{f_{i+1, j} - 2f_{i, j} + f_{i-1, j}}{\Delta x^2}$$

$$\Delta\phi(x, y) = 0$$

$$\frac{f_{i+1, j} - 2f_{i, j} + f_{i-1, j}}{\Delta x^2} + \frac{f_{i, j+1} - 2f_{i, j} + f_{i, j-1}}{\Delta y^2} = 0$$

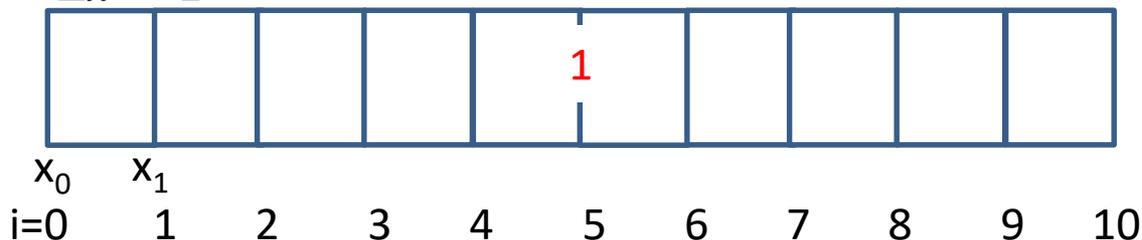
後は境界条件

$\frac{d^2\phi(x)}{dx^2} = q(x)$  の1次元Poisson 方程式で  $x=0, 10$  で  $\phi(x)=0$ ,  $q(x) = 1$  at  $x = 5$   
 $0$  at  $x \neq 5$

の場合の  $\phi(x)$  を求める

差分化 
$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} = q_i$$

$\Delta x = 1$



$$\phi_2 = 2\phi_1 - \phi_0$$

$$\phi_3 = 2\phi_2 - \phi_1$$

$$\phi_4 = 2\phi_3 - \phi_2$$

$$\phi_5 = 2\phi_4 - \phi_3 =$$

$$\phi_6 = 2\phi_5 - \phi_4 + 1$$

$$\phi_7 = 2\phi_6 - \phi_5$$

$$\phi_8 = 2\phi_7 - \phi_6$$

$$\phi_9 = 2\phi_8 - \phi_7$$

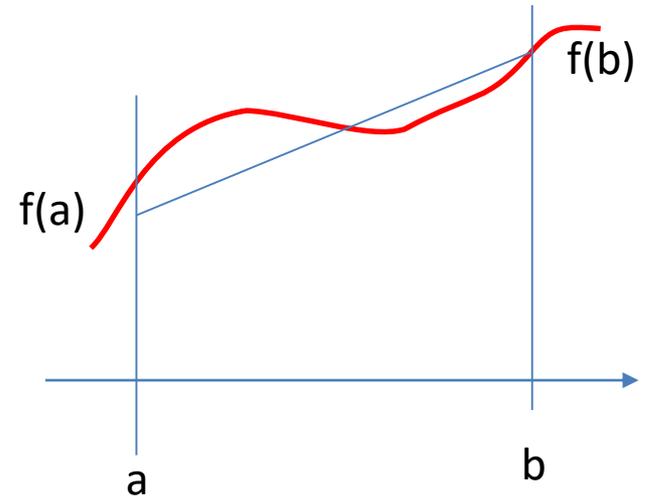
$$\phi_{10} = 2\phi_9 - \phi_8$$

9個の方程式

9個の未知数

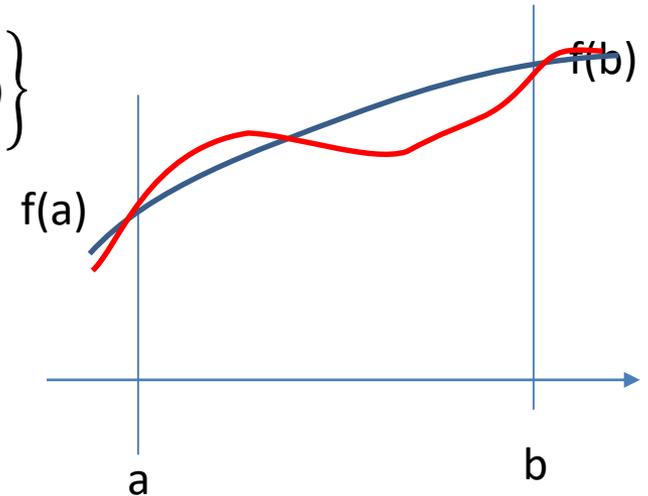
## 積分の公式

$$\int_a^b f(x) dx = (b - a) \frac{f(a) + f(b)}{2}$$



## シンプソンの公式

$$\int_a^b f(x) dx = \frac{(b - a)}{6} \left\{ f(a) + 4f\left(\frac{a + b}{2}\right) + f(b) \right\}$$

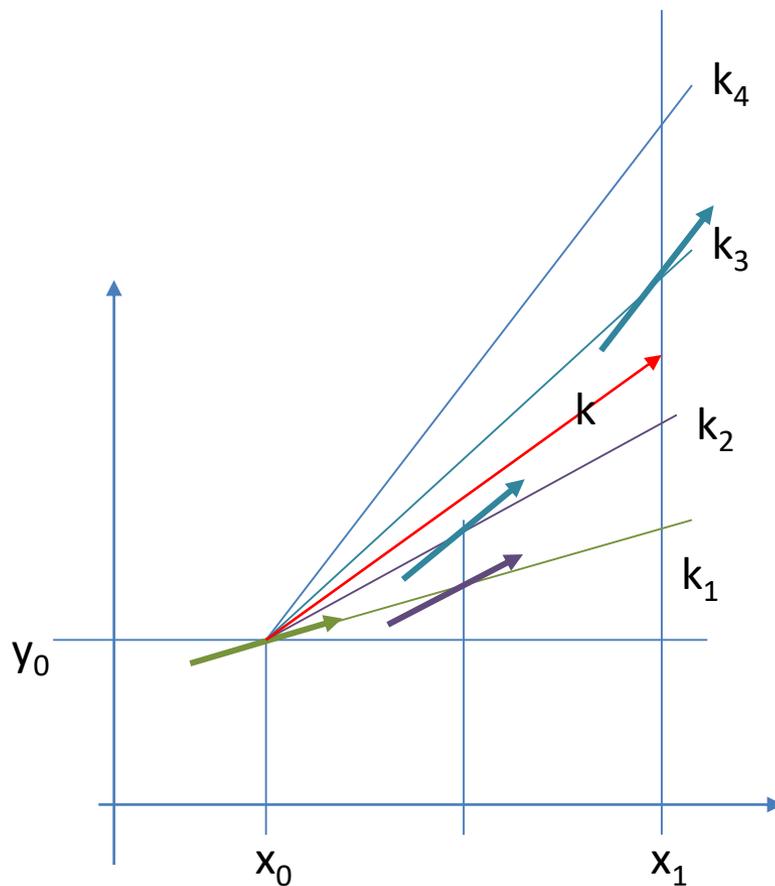


$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$  前進積分では、誤差が加算されて行ってしまう。

より精度の高い差分近似による積分  $\rightarrow$  Runge - Kutta 法

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{aligned} k_1 &= f(x_0, y_0)h \\ k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)h \\ k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)h \\ k_4 &= f(x_0 + h, y_0 + k_3)h \end{aligned}$$



$$\frac{dy}{dt} = -y, \quad y(0) = 1$$

の時、 $\Delta t = 0.2$ とした場合の $0 \leq t \leq 1$ の解をRunge-Kutta法で求める。

t	y	f = -y	h	ki	k
0	1	-1	0.2	-0.2	
0.1	0.9	-0.9	0.2	-0.18	
0.1	0.91	-0.91	0.2	-0.182	-0.18127
0.2	0.818	-0.818	0.2	-0.1636	
0.2	0.818733	-0.81873	0.2	-0.16375	
0.3	0.73686	-0.73686	0.2	-0.14737	
0.3	0.745047	-0.74505	0.2	-0.14901	-0.14841
0.4	0.669724	-0.66972	0.2	-0.13394	
0.4	0.670324	-0.67032	0.2	-0.13406	
0.5	0.603292	-0.60329	0.2	-0.12066	
0.5	0.609995	-0.61	0.2	-0.122	-0.12151
0.6	0.548325	-0.54833	0.2	-0.10967	
0.6	0.548817	-0.54882	0.2	-0.10976	
0.7	0.493935	-0.49394	0.2	-0.09879	
0.7	0.499423	-0.49942	0.2	-0.09988	-0.09948
0.8	0.448932	-0.44893	0.2	-0.08979	
0.8	0.449335	-0.44933	0.2	-0.08987	
0.9	0.404401	-0.4044	0.2	-0.08088	
0.9	0.408895	-0.40889	0.2	-0.08178	-0.08145
1	0.367556	-0.36756	0.2	-0.07351	

0.367885

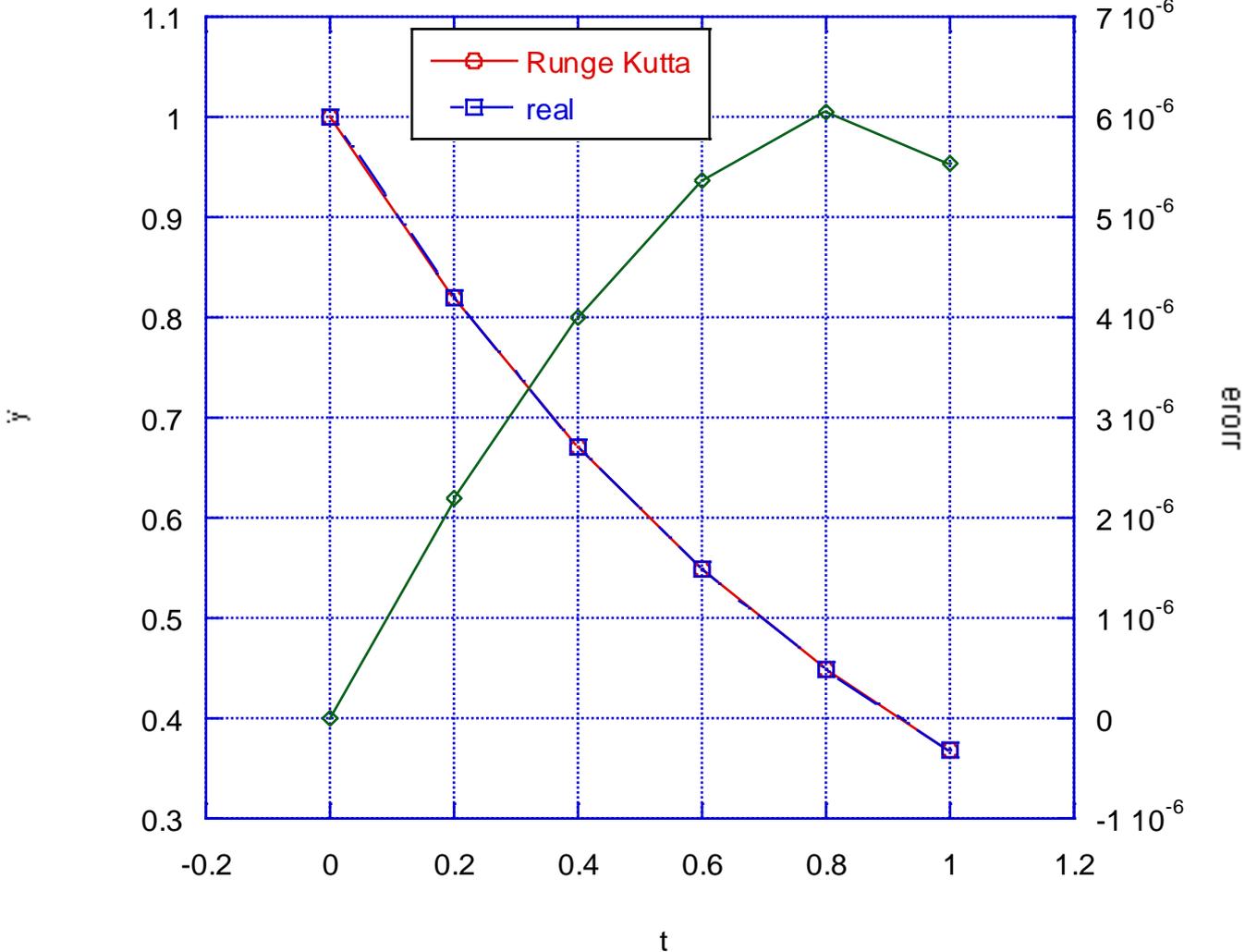
$$\frac{dy}{y} = -dt$$

$$\ln(y) = -t + C$$

$$y = y_0 e^{-t}$$

D

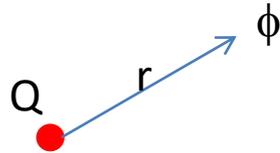
Data 1



# モーメント法

点電荷

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

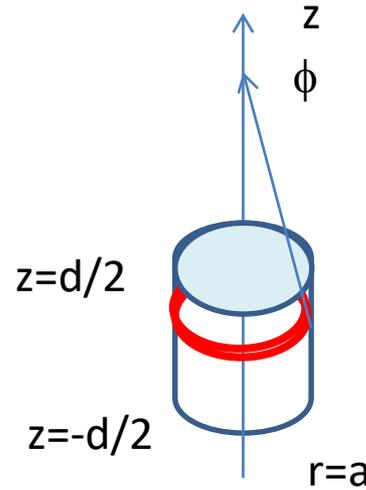


半径a、長さdの一様な電荷の場合

$$\begin{aligned} \phi &= \frac{1}{4\pi\epsilon_0} \int_{-d/2}^{d/2} \int_0^{2\pi} \frac{\sigma}{R} a d\varphi' dz' \\ &= \frac{2\pi\sigma a}{4\pi\epsilon_0} \int_{-d/2}^{d/2} \frac{1}{\sqrt{a^2 + (z - z')^2}} dz' \\ &= \frac{\sigma a}{2\epsilon_0} \ln \frac{z + \frac{d}{2} + \sqrt{a^2 + (z + d/2)^2}}{z - \frac{d}{2} + \sqrt{a^2 + (z - d/2)^2}} = \sigma f(z) \end{aligned}$$

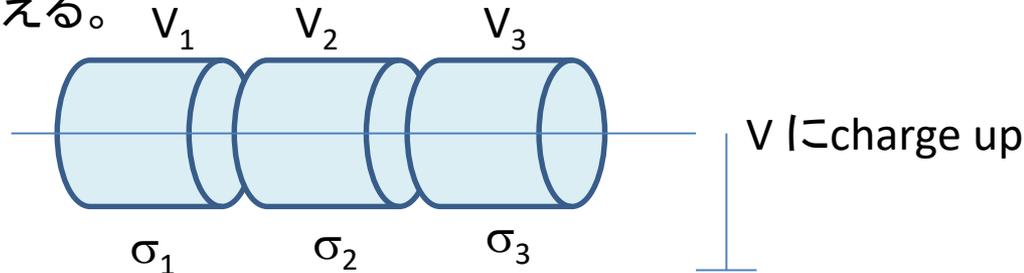
リング状の電荷

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(r')}{R} ds'$$



# モーメント法 (cont.)

3分割で考える。



$$V_1 = \sigma_1 f(0) + \sigma_2 f(d) + \sigma_3 f(2d)$$

$$V_2 = \sigma_1 f(d) + \sigma_2 f(0) + \sigma_3 f(d)$$

$$V_3 = \sigma_1 f(2d) + \sigma_2 f(d) + \sigma_3 f(0)$$

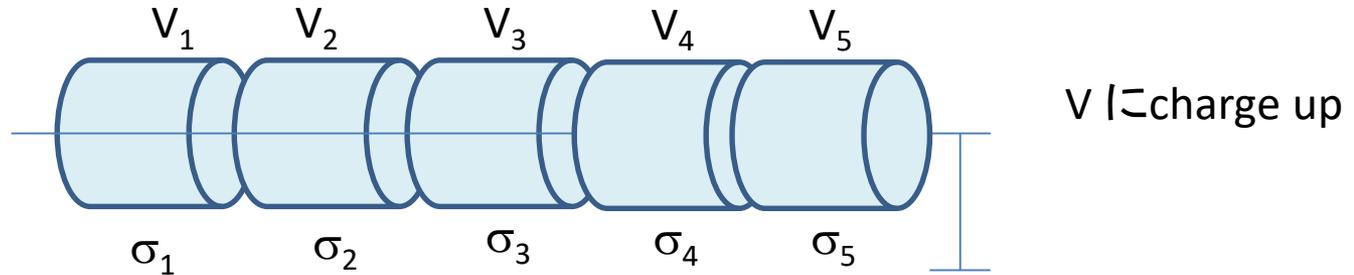
電位が  $V_1, V_2, V_3$  とも  $V$  であるなら

この逆行列を解けばいい

$$\begin{pmatrix} f(0) & f(d) & f(2d) \\ f(d) & f(0) & f(d) \\ f(2d) & f(d) & f(0) \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = V \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

# モーメント法 (cont.)

5になれば



$$V_1 = \sigma_1 f(0) + \sigma_2 f(d) + \sigma_3 f(2d) + \sigma_4 f(3d) + \sigma_5 f(4d)$$

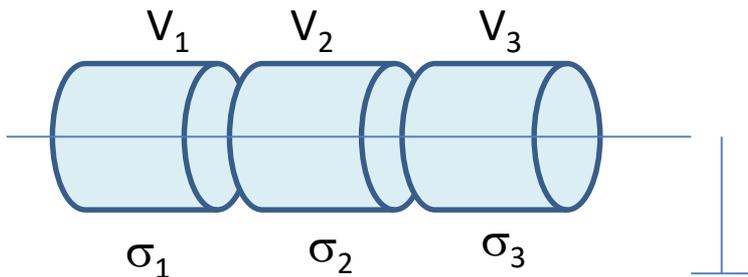
$$V_2 = \sigma_1 f(d) + \sigma_2 f(0) + \sigma_3 f(d) \dots$$

$$V_3 = \sigma_1 f(2d) + \sigma_2 f(d) + \sigma_3 f(0) \dots \dots$$

.....

この逆行列を解けばいい

$$(5 \times 5)(5) = (5)$$



$a = 1\text{mm}$   
 $d = 1\text{m}$ とすると

[13.81551256, 1.0986114, 0.510825527]  
 [1.0986114, 13.81551256, 1.0986114]  
 [0.510825527, 1.0986114, 13.81551256]

f(d) matrix

[0.07291239861, -0.005619156201, -0.002249083928]  
 [-0.005619156201, 0.07327607527, -0.005619156201]  
 [-0.002249083928, -0.005619156201, 0.07291239861]

逆行列

0.06504415848, 0.06203776287, 0.06504415848

$\sigma$ の分布

